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ARITHMETIC

AND

ALGEBRA,

COMBINING THE MOST IMPORTANT PARTS OF THOSE SUBJECTS,

WITH

A VARIETY OF EXAMPLES.

1842.

THE PRINCIPLES

OF

ARITHMETIC,

CONTAINING

A VARIETY OF EXAMPLES FOR PRACTICE,

WITH A SUFFICIENT NUMBER WORKED AT LENGTH TO SHEW THE SOLUTION OF EVERY DIFFICULTY ANTICIPATED.

BY

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ADVERTISEMENT.

On the publication of the Second Edition of his Arithmetic and Algebra, the Author has only to observe, that he has re-written whatever appeared to be too concise or difficult for the student, and that he has added a great number and variety of examples for practice.

N.B. The Arithmetical and Algebraical portion of the Schedule of Mathematical subjects for Examination for the Degree of B. A. will be found at the end of the Algebra.

CONTENTS.

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ARITHMETIC.

Art. 1. In order to estimate the relations of quantities by means of numbers, which is the object of the science termed *Arithmetic*, it is necessary to fix upon some definite portion as a standard.

This standard, which is the foundation of our various calculations, is denominated *Unit:* thus, a second may be the unit of time, an inch or a foot may be the unit of length, a pound that of weight, and so on for any denomination of things or their qualities.

2. An unit, or the aggregate of any number of units, is called an *Integer* or *Whole Number*.

3. The expressing of numbers by figures, or *Digits* (as they are called, from the old custom of reckoning by the fingers), is termed *Notation;* and the reading, or estimating of numbers so expressed, *Numeration*.

4. The *common system of Notation*, or that in which the local values of the figures is ten times their simple value, for every place they are removed to the left of the place of units; together with the fundamental operations of *Addition*, *Subtraction*, *Multiplication*, and *Division of Integers*, is here presumed to be understood, being commonly taught orally.

5. The operations of Arithmetic, in many cases, are aided by *signs* or *abbreviations*, some of which are the following:—

$\therefore$ is an abbreviation signifying *therefore.*

The sign $=$ signifies that the two quantities between which it is placed are *equal.*

The sign $+$, called *plus*, and $-$, called *minus*, indicate respectively the inverse operations of addition and subtraction: thus, $9 + 3 = 12$, and $12 - 3 = 9$.

The sign $\times$, *into*, and $\div$, *by*, indicate respectively those of multiplication and division: thus, $9 \times 3 = 27$, and $27 \div 3 = 9$.

Division is also indicated by writing the dividend over the divisor, with a line between them; in which case the expression is called a *Fraction.*

FRACTIONS.

6. As an integer denotes an unit or the aggregate of any number of units, so a fraction represents a part or any number of equal parts into which an unit is divided.

7. The integer below the line, or the divisor, denoting the number of parts into which the unit is divided, and therefore indicating their value compared with the unit, is termed the *Denominator:* that above the line, or the dividend, shewing how many such parts are taken, is called the *Numerator.*

8. When the numerator is less than the denominator, the fraction is called a *Proper Fraction*, as $\frac{3}{4}$, $\frac{5}{8}$, &c.

9. When the numerator is equal to, or greater than the denominator, it is called an *Improper Fraction*, as $\frac{3}{3}$, $\frac{7}{5}$, &c.

10. A quantity consisting of a whole number and a fraction is called a *Mixed Number*, as $6\frac{3}{8}$, which represents 6 units together with $\frac{3}{8}$ of the unit.

11. *To multiply a fraction by any number.*

Multiply the numerator by that number, and retain the same denominator.

$$\frac{4}{15} \times 2 = \frac{8}{15};$$

for the unit in each of the fractions $\frac{4}{15}$ and $\frac{8}{15}$ is divided into 15 equal parts, and the number of those parts taken in the latter is twice the number in the former fraction.

Or, when the denominator of the fraction is divisible by the given number, divide it by that number; for the fraction will thereby be multiplied by the given number, and the product expressed in lower terms.

$$\frac{4}{15} \times 3 = \frac{4}{5};$$

for, the same number of parts is taken in both of the fractions $\frac{4}{15}$ and $\frac{4}{5}$, and the unit being divided into three times as many equal parts in $\frac{4}{15}$ as in $\frac{4}{5}$, each of the parts in the latter is three times each in the former; therefore the latter fraction is equal to three times the former.

12. *To divide a fraction by a given number.*

Multiply the denominator by that number, and retain the same numerator.

$$\frac{3}{4} \div 4 = \frac{3}{16};$$

for the same number of parts is taken in both of the frac-

tions $\frac{3}{4}$ and $\frac{3}{16}$, and the unit being divided into four times as many equal parts in the latter as in the former fraction, each of the parts in the latter is one-fourth of each in the former; therefore the latter fraction is one-fourth of the former.

Or, when the numerator of the fraction is divisible by the given number, divide it by that number, for the fraction will thereby be divided by the given number, and the quotient expressed in lower terms.

$$\frac{9}{10} \div 3 = \frac{3}{10};$$

for the unit in $\frac{9}{10}$ and $\frac{3}{10}$ is divided into the same number of equal parts, and the number of those parts taken in the latter is one-third the number in the former; therefore the latter fraction is one-third of the former.

13. *If both the numerator and denominator of a fraction be multiplied or divided by the same number, the value of the fraction is not altered.*

For, if the numerator be multiplied by any number, the fraction is multiplied by that number; and if the denominator be multiplied by the same number, the fraction is divided by it: or, if the numerator be divided by any number, the fraction is divided by that number; and if the denominator be divided by the same number, the fraction is multiplied by it. That is, if both the numerator and denominator of a fraction be either multiplied or divided by the same number, the fraction is both multiplied and divided by that number, and therefore its value is not thereby altered.

$$\frac{3}{8} = \frac{3 \times 3}{8 \times 3} = \frac{9}{24} = \frac{9 \times 10}{24 \times 10} = \frac{90}{240},$$

$$\text{or } \frac{90}{240} = \frac{90 \div 10}{240 \div 10} = \frac{9}{24} = \frac{9 \div 3}{24 \div 3} = \frac{3}{8}.$$

COROLLARY. Hence, multipliers, or *Factors*, as they are frequently called, which are common to both the numerator and denominator, may be cancelled, and the fractions or fractional expressions will thus be reduced to lower and more simple terms.

14. *To express an integer as a fraction having a given denominator.*

Multiply the proposed number by the given denominator, and the product will be the numerator of the fraction required. Thus, 306 expressed as a fraction whose denominator is 7, is

$$\frac{306 \times 7}{7} \text{ or } \frac{2142}{7};$$

for, since each unit is supposed to consist of 7 equal parts, 2142 or 7 times 306 of those parts are equal to 306 units.

In the same manner 3, 6, 9, 12, 15, and 5, written as fractions whose denominators are respectively 4, 7, 10, 13, 16, and 1, are

$$\frac{12}{4}, \frac{42}{7}, \frac{90}{10}, \frac{156}{13}, \frac{240}{16} \text{ and } \frac{5}{1}.$$

15. *To reduce an improper fraction to its equivalent whole or mixed number.*

Divide the numerator by the denominator, and the quotient will be the whole number; and if there be a remainder, place it over the denominator, and the fraction thus formed annexed to the whole number will be the mixed number required.

$$\frac{34}{5} = 6\frac{4}{5};$$

for the unit being divided into 5 equal parts, and 34 such parts being taken, there are 6 units and 4 such parts.

In the same manner the mixed or whole numbers respectively corresponding to the fractions

$$\frac{9}{8}, \frac{15}{3}, \frac{21}{16}, \frac{99}{20}, \frac{136}{7} \text{ and } \frac{72}{6},$$

are $1\frac{1}{8}$, 5, $1\frac{5}{16}$, $4\frac{19}{20}$, $19\frac{3}{7}$ and 12.

16. *To find the greatest common factor of any two numbers, and to reduce a fraction to its lowest terms.*

If the numerator and denominator of a fraction be divided by any factor common to both, as in Art. 13, the fraction is reduced to lower terms; and if this process be repeated until there is no factor common to both, the fraction is reduced to its lowest terms. But if they be divided by their greatest common factor, the fraction will be reduced at once to its lowest terms. This factor the multiplication table will in many cases suggest. Thus the greatest common factor of 15 and 35 is 5, and

$$\frac{15}{35} = \frac{15 \div 5}{35 \div 5} = \frac{3}{7}.$$

When the numerator or denominator, or both of them, exceed the limits of the multiplication table, other means of ascertaining the greatest common factor must be used. For instance, let it be required to find the greatest common factor of 78 and 324, in order to reduce the fraction $\frac{78}{324}$ to its lowest terms. Dividing the greater by the less, we have

$$324 = 78 \times 4 + 12;$$

whence every common factor of 324 and 78 is also a factor of 12; for every factor of 78 is a factor of 78×4; and if a number be resolved into two parts, and a factor of the whole

be contained any number of times in one part without a remainder, it must also be contained a certain number of times in the other, the sum of the times it is contained in the two parts being equal to the number of times it is contained in the whole: so that the operation is thus reduced to that of finding the greatest common factor of 78 and 12; and, dividing the greater of these by the less, we have

$$78 = 12 \times 6 + 6,$$

therefore, reasoning as above, every common factor of 78 and 12 is also a common factor of 12 and 6; now the greatest common factor of 12 and 6 is obviously 6, therefore 6 is the factor required; and

$$\frac{78}{324} = \frac{78 \div 6}{324 \div 6} = \frac{13}{54},$$

which is the given fraction reduced to its lowest terms.

The operation of finding the greatest common factor is usually written thus,

```
78) 324 (4
    312
    ---
     12) 78 (6
         72
         --
          6) 12 (2
             12
             --
             ...
```

The nature of the process for finding the greatest common factor being the same whatever be the fraction proposed, the steps above taken suggest the following general rule:

Divide the greater by the less, and the divisor by the remainder continually, until there is no remainder; the last divisor is the factor required.

Ex. 1. Find the greatest common factor of 3860 and 4768, and reduce the fraction $\frac{3860}{4768}$ to its lowest terms.

$$
\begin{array}{l}
3860)\ 4768\ (1 \\
\qquad\quad 3860 \\
\qquad\quad \overline{\ \ 908)\ 3860\ (4} \\
\qquad\qquad\qquad 3632 \\
\qquad\qquad\qquad \overline{\ \ 228)\ 908\ (3} \\
\qquad\qquad\qquad\qquad\quad 684 \\
\qquad\qquad\qquad\qquad\quad \overline{224)\ 228\ (1} \\
\qquad\qquad\qquad\qquad\qquad\quad 224 \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{\ \ \ 4)\ 224\ (56} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 224 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\ \ldots\ }
\end{array}
$$

Hence 4 is the greatest common factor; and

$$\frac{3860}{4768} = \frac{3860 \div 4}{4768 \div 4} = \frac{965}{1192},$$

which is the fraction in its lowest terms.

Ex. 2. Reduce the following fractions to their lowest terms:

$$\frac{856}{936}, \frac{3723}{46320}, \frac{36425}{978910}. \qquad \textit{Ans.}\ \frac{107}{117}, \frac{1241}{15440}, \frac{7285}{195782}.$$

Cor. 1. In the above explanation of the process for finding the greatest common factor of two numbers, it is shewn that every common factor of the given numbers is also a factor of each remainder; whence it is evident that every common divisor of any two numbers is also a divisor of their difference. Thus, let the numbers be 509 and 524; their difference is 15, the only divisors of which are 3 and 5, and neither of these is a divisor of the given numbers: therefore 509 and 524 have not a common divisor.

COR. 2. Whenever we arrive, in the successive divisions, at a remainder which is not composed of factors, we may conclude that the given numbers either have no common factor, or that this remainder is their greatest common factor.

The following are some numbers not composed of factors, called *Prime Numbers :*

2	19	47	79	433	2749
3	23	53	83	541	3389
5	29	59	89	647	4673
7	31	61	97	761	5189
11	37	67	179	857	7321
13	41	71	257	953	9967
17	43	73	397	1759	18583

From these numbers examples may be made: thus, take any two, say 53 and 179, and multiply both by any number, say 9, giving 477 and 1611; the greatest common factor of these numbers is 9, and

$$\frac{477}{1611}, \text{ reduced to its lowest terms, gives } \frac{53}{179}.$$

17. *To find the greatest common factor of three or more quantities.*

Find the greatest common factor of the first two, then the greatest common factor of this and the third, which will be the greatest common factor of the first three; then find the greatest common factor of this and the fourth, and so on for any number of quantities.

Ex. 1. Find the greatest common factor of 36, 90, 729 and 3603.

The greatest common factor of 36 and 90 is 18, of 18 and 729 is 9, and of 9 and 3603 is 3; therefore 3 is the factor required.

Ex. 2. Find the greatest common factor of 12, 16, and 18.

Ans. 2.

Ex. 3. Find the greatest common factor of 28, 84, 154, and 343. *Ans.* 7.

Ex. 4. Find the greatest common factor of 377, 1079, 247, and 689. *Ans.* 13.

Ex. 5. Find the greatest common factor of 289, 799, 2533, and 3077. *Ans.* 17.

Ex. 6. Find the greatest common factor of 160104, 461832, and 871736. *Ans.* 168.

More examples may be made from the prime numbers, as in the preceding article.

18. By means of the following *Properties of Numbers*, common factors may frequently be discovered, and fractions readily reduced thereby to their lowest terms.

A number is divisible by

Two, when the right-hand digit is divisible by two, or even, as 26, 58, 166.

Three or *Nine*, when the sum of its digits is divisible by three or nine respectively: as 123, 279, in which $1+2+3$ or 6 is divisible by 3, and $2+7+9$ or 18 is divisible by 3 or 9.

Four, when the number consisting of the two right-hand digits is divisible by four: as 1084, in which 84 is divisible by 4.

Five, when the right-hand digit is either 0 or 5: as 830, 295.

Six, when it is divisible by *two* (or even) and by *three:* as 138.

Eight, when the number consisting of the three right-hand digits is divisible by eight: as 2936, in which 936 is divisible by 8.

Ten, when the right-hand digit is a cipher: as 9530.

Eleven, when the two sums found by taking alternate digits either are equal, or differ by a number which is divisible by eleven: as 926519, in which $9+5+2=1+6+9$, and 381909, in which $9+9+8$ differs from $0+1+3$ by 22, which is divisible by 11.

Twelve, when it is divisible by *four* and *three*: as 996.

All *prime* numbers, excepting 2 and 5, have 1, 3, 7, or 9 in the place of units; but the converse is not true: all other numbers are *composite*, or composed of factors.

EXAMPLES.

(1) Reduce $\frac{394}{672}$ to its lowest terms. *Ans.* $\frac{197}{336}$.

(2) Reduce $\frac{2382}{2919}$ and $\frac{4131}{7452}$ to their lowest terms.

Ans. $\frac{794}{973}$ and $\frac{51}{92}$.

(3) Reduce $\frac{1312}{2288}$ to its lowest terms. *Ans.* $\frac{82}{143}$.

(4) Reduce $\frac{775}{1800}$ to its lowest terms. *Ans.* $\frac{31}{72}$.

(5) Reduce $\frac{138}{174}$ to its lowest terms. *Ans.* $\frac{23}{29}$.

(6) Reduce $\frac{27968}{37376}$ to its lowest terms. *Ans.* $\frac{437}{584}$.

(7) Reduce $\frac{3700}{5900}$ to its lowest terms. *Ans.* $\frac{37}{59}$.

(8) Reduce $\frac{3872}{92807}$ to its lowest terms. *Ans.* $\frac{32}{767}$.

(9) Reduce $\frac{6492}{19644}$ to its lowest terms. *Ans.* $\frac{541}{1637}$.

19. When the numerator of a fractional expression, which consists of numbers with the sign of addition or subtraction

between them, is to be divided, each of the numbers must be divided.

$$\frac{34 + 8 - 4}{2} = 17 + 4 - 2 = 19.$$

But if the numbers have the sign of multiplication between them, only one of them may be divided. (See Art. 13.)

$$\frac{6 \times 4 \times 12}{3} = 2 \times 4 \times 12 \text{ or } 6 \times 4 \times 4 = 96.$$

20. *To transform any number of fractions having different denominators to other equivalent fractions with a common denominator.*

This may be done by multiplying the numerator and denominator of each fraction by all the denominators except its own. The product of all the denominators will thus be made the common denominator, and, by Art. 10, the respective values of the given fractions will not thereby be altered.

Ex. 1. Transform $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{7}$ to equivalent fractions, having a common denominator,

$$\frac{1}{2} = \frac{1 \times 3 \times 7}{2 \times 3 \times 7} = \frac{21}{42}$$

$$\frac{1}{3} = \frac{1 \times 2 \times 7}{3 \times 2 \times 7} = \frac{14}{42}$$

$$\frac{2}{7} = \frac{2 \times 2 \times 3}{7 \times 2 \times 3} = \frac{12}{42}.$$

The operation may be written thus,

$1 \times 3 \times 7 = 21$, new numerator for $\frac{1}{2}$,

$1 \times 2 \times 7 = 14$, ... $\frac{1}{3}$,

$2 \times 2 \times 3 = 12$, ... $\frac{2}{7}$,

$2 \times 3 \times 7 = 42$, common denominator;

whence the required fractions are $\frac{21}{42}, \frac{14}{42}, \frac{12}{42}$.

Ex. 2. Transform $\frac{3}{8}$ and $\frac{2}{9}$ to equivalent fractions with a common denominator. *Ans.* $\frac{27}{72}$ and $\frac{16}{72}$.

Ex. 3. Transform $\frac{4}{5}, \frac{5}{7}, \frac{7}{9}$ to equivalent fractions with a common denominator. *Ans.* $\frac{252}{315}, \frac{225}{315}, \frac{245}{315}$.

Ex. 4. Transform $\frac{1}{2}, \frac{1}{3}, \frac{3}{5}, \frac{2}{19}$ to equivalent fractions with a common denominator. *Ans.* $\frac{285}{570}, \frac{190}{570}, \frac{342}{570}, \frac{60}{570}$.

In many cases the fractions found by this method may be reduced to lower terms, and still have a common denominator; which may be done by dividing all the new numerators and the common denominator by any common measure: and if they be divided by their greatest common measure, the resulting fractions will have the least possible common denominator.

Ex. Transform $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ to equivalent fractions having the least common denominator.

$$\frac{1}{2} = \frac{1 \times 3 \times 4}{2 \times 3 \times 4} = \frac{12}{24},$$

$$\frac{1}{3} = \frac{1 \times 2 \times 4}{3 \times 2 \times 4} = \frac{8}{24},$$

$$\frac{1}{4} = \frac{1 \times 2 \times 3}{4 \times 2 \times 3} = \frac{6}{24}.$$

and the greatest common measure of 12, 8, 6, and 24 is 2;

$$\therefore \text{ the required fractions are } \frac{6}{12}, \frac{4}{12}, \frac{3}{12}.$$

In this example, 12, the required common denominator, is the least number which exactly contains each of the denominators of the given fractions, and is therefore called their *Least Common Multiple:* and this being the case in any example, *another method*, is to find the least common multiple of the denominators, and then to transform the fractions into others of equal value, having the least common multiple of the given denominators for their common denominator, by the two following articles.

21. *To find the Least Common Multiple of any two or more numbers.*

Ex. 1. Suppose the least common multiple of 6 and 8 to be required: then, by resolving these numbers into their factors, thus,

$$6 = 2 \times 3, \text{ and } 8 = 2 \times 4,$$

it appears that the least number which is divisible by 6 and 8, without a remainder, is $2 \times 3 \times 4$, or 24, found by rejecting from one of them the factor found in both, and taking the product of the remaining factors; which is in effect dividing the product of the two numbers by their common factor, 2: for

$$\frac{6 \times 8}{2} = \frac{2 \times 3 \times 2 \times 4}{2} = 2 \times 3 \times 4 = 24.$$

Ex. 2. Let the proposed numbers be 24 and 30: then

$$24 = 2 \times 3 \times 4, \text{ and } 30 = 2 \times 3 \times 5;$$

whence it appears that the least number exactly divisible by them is $2 \times 3 \times 4 \times 5$, or 120, found by rejecting from one of them the common factors, and taking the product of the remaining factors; which is the same thing as dividing the product of the two numbers by the product of their common

factors, or by their greatest common factor (Art. 16), thus

$$\frac{24\times 30}{6}=\frac{2\times 3\times 4\times 2\times 3\times 5}{2\times 3}=2\times 3\times 4\times 5=120;$$

hence the following general rule.

RULE. To find the least common multiple of two numbers, divide their product by their greatest common measure, and the quotient will be their least common multiple.

If there be more than two numbers, proceed in the same manner with the least common multiple of any two of them and a third number, and so on until they are all taken: the last quotient will be the least common multiple sought.

Ex. 3. Find the least common multiple of 4, 9, 18, and 45.

4 and 9 have not a common factor,

$\therefore$ their least common multiple is $4\times 9=36$;

the least com. mult. of 36 and 18 $=\dfrac{36\times 18}{18}=36$,

........................36 and 45 $=\dfrac{36\times 45}{9}=4\times 45=180$;

whence 180 is the least number which is exactly divisible by each of the given numbers.

Ex. 4. Find the least common multiple of 3, 7, and 9.

Ans. 63.

Ex. 5. Find the least common multiple of 5, 9, 10, and 15.

Ans. 90.

22. *To transform fractions with different denominators to others of equal value, having the least common multiple of their denominators for a common denominator.*

Find the least common multiple of the denominators of the given fractions, divide it by the denominator of each fraction, and multiply the corresponding numerator by the

quotient: the products thence arising will be the numerators of the fractions required.

The proposed fractions will thus, as in Art. 20, have their corresponding numerators and denominators multiplied by the same numbers: consequently their respective values will not be altered.

Ex. 1. Effect the necessary transformation that $\frac{3}{4}$ and $\frac{7}{12}$ may have the least common multiple of their denominators for a common denominator.

Here 12 is obviously the least common multiple of the given denominators; whence

$$\frac{12}{4} = 3, \text{ and } \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}.$$

Ex. 2. Transform $\frac{1}{2}$, $\frac{5}{6}$, and $\frac{7}{10}$ to equivalent fractions, having the least common multiple of their denominators for their common denominator.

The least common multiple

$$\text{of } 2 \text{ and } 6 = \frac{2 \times 6}{2} = 6,$$

$$\text{of } 6 \text{ and } 10 = \frac{6 \times 10}{2} = 3 \times 10 = 30;$$

whence

$$\frac{30}{2} = 15, \text{ and } \frac{1}{2} = \frac{15}{2 \times 15} = \frac{15}{30},$$

$$\frac{30}{6} = 5, \text{ and } \frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30},$$

$$\frac{30}{10} = 3, \text{ and } \frac{7}{10} = \frac{7 \times 3}{10 \times 3} = \frac{21}{30}.$$

Ex. 3. Express $\frac{1}{3}$ and $\frac{4}{27}$ with the least common denominator.

Ans. $\frac{9}{27}$ and $\frac{4}{27}$.

Ex. 4. Transform $\frac{2}{9}$ and $\frac{7}{15}$ to equivalent fractions having the least common denominator. *Ans.* $\frac{10}{45}$, $\frac{21}{45}$.

Ex. 5. Transform $\frac{3}{8}$, $\frac{9}{20}$, $\frac{13}{24}$ to equivalent fractions, having the least common denominator.

Ans. $\frac{45}{120}$, $\frac{54}{120}$, $\frac{65}{120}$.

Ex. 6. Transform $\frac{2}{9}$, $\frac{4}{15}$, $\frac{7}{18}$, and $\frac{12}{25}$ to equivalent fractions with the least common multiple of their denominators, for a common denominator.

Ans. $\frac{100}{450}$, $\frac{120}{450}$, $\frac{175}{450}$, $\frac{216}{450}$.

23. *To compare the values of fractions.*

(*a*) When fractions have a common denominator, they have the relative values of their numerators.

$$\frac{2}{5} \text{ is the half of } \frac{4}{5}; \text{ for } \frac{2}{5} \times 2 = \frac{4}{5} \text{ (Art. 11).}$$

Hence, the values of fractions having different denominators may be compared by transforming them to equivalent fractions with a common denominator.

Ex. Compare the values of $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{17}$.

$\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{17}$ are respectively equal to $\frac{85}{255}$, $\frac{102}{255}$, and $\frac{45}{255}$, and, therefore, have the relative values of 85, 102, and 45, the numerators of the transformed fractions.

(*b*) When fractions have a common numerator, they have inversely the relative values of their denominators, the greater having the less denominator.

$$\frac{5}{7} \text{ is greater than } \frac{5}{8}.$$

For the same number of parts is taken in both, and the unit being divided into more equal parts in the latter than in the former fraction, each of the parts in the former is greater than each in the latter.

ADDITION OF FRACTIONS.

24. (*a*) The sum of any number of fractions having a common denominator, is found by taking the sum of the numerators, and subjoining the common denominator.

For, since the unit is divided into the same number of equal parts in each of the fractions, and the several numerators denote the number of those parts in each, the sum of the fractions is the sum of the numerators with the common denominator subjoined.

Ex. 1. $\frac{3}{13} + \frac{5}{13} = \frac{3+5}{13} = \frac{8}{13}$.

Here the unit is divided into thirteen equal parts, and to three of those parts five are added, the sum is, therefore, eight such parts, *i.e.* $\frac{8}{13}$.

(*b*) If the fractions have not a common denominator, transform them to others of equal value with a common denominator, and proceed as before.

When the least common multiple of the denominators is obvious, or easily discovered, the transformation by Art. 22 will be found convenient.

Ex. 2. Required the sum of $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{4}{7}$.

By Art. 20,

$$2 \times 5 \times 7 = 70, \text{ new numerator for } \frac{2}{3},$$

$$3 \times 3 \times 7 = 63, \text{ .... } \frac{3}{5},$$

$$4 \times 3 \times 5 = 60, \text{ .... } \frac{4}{7},$$

$$\overline{193} \text{ Total,}$$

and $3 \times 5 \times 7 = 105$ common denominator;

$$\therefore \frac{2}{3} + \frac{3}{5} + \frac{4}{7} = \frac{193}{105} = 1\frac{88}{105}.$$

Ex. 3. Required the sum of $\frac{1}{4}$, $\frac{5}{6}$, and $\frac{3}{8}$.

The least common multiple of the denominators is obviously 24; whence, by Art. 22,

$$\frac{1}{4} = \frac{6}{24}, \quad \frac{5}{6} = \frac{5 \times 4}{24} = \frac{20}{24}, \quad \frac{3}{8} = \frac{3 \times 3}{24} = \frac{9}{24};$$

$$\therefore \frac{1}{4} + \frac{5}{6} + \frac{3}{8} = \frac{6 + 20 + 9}{24} = \frac{35}{24} = 1\frac{11}{24}.$$

Ex. 4. Required the sum of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{5}{6}$.

Here, by inspection, it is obvious that 30 is the least common multiple of the denominators; whence

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{5} + \frac{5}{6} = \frac{15+20+18+25}{30} = \frac{78}{30} = \frac{13}{5} = 2\frac{3}{5}.$$

Ex. 5. Required the sum of $\frac{1}{3}$, $\frac{3}{8}$, and $\frac{5}{12}$. *Ans.* $1\frac{1}{8}$.

Ex. 6. Required the sum of $\frac{1}{2}$, $\frac{3}{8}$, and $\frac{5}{19}$. *Ans.* $1\frac{21}{152}$.

Ex. 7. Required the sum of $\frac{7}{15}$, $\frac{10}{21}$, and $\frac{16}{35}$ *Ans.* $1\frac{2}{5}$.

Ex. 8. Required the sum of $\frac{1}{3}$, $\frac{2}{9}$, $\frac{5}{36}$, and $\frac{15}{44}$.

Ans. $1\frac{7}{198}$.

(*c*) When mixed numbers are to be added, the sum of the fractions should be taken as before, and annexed to the sum of the integers.

Ex. 9. Required the sum of $3\frac{1}{7}$, $5\frac{3}{14}$, $12\frac{7}{9}$, and $24\frac{5}{12}$.

$$3\frac{1}{7}+5\frac{3}{14}+12\frac{7}{9}+24\frac{5}{12}=3+5+12+24+\frac{1}{7}+\frac{3}{14}+\frac{7}{9}+\frac{5}{12}$$

$$=44+\frac{391}{252}=45\frac{139}{252}.$$

Ex. 10. Required the sum of $4\frac{2}{9}$, $6\frac{5}{13}$, and $7\frac{8}{19}$.

Ans. $18\frac{62}{2223}$.

Ex. 11. Required the sum of $17\frac{5}{12}$, $\frac{4}{15}$, and $144\frac{11}{21}$.

Ans. $162\frac{29}{140}$.

Ex. 12. Required the sum of $1\frac{1}{2}+2\frac{2}{3}+3\frac{3}{4}+4\frac{4}{5}+5\frac{5}{6}$.

Ans. $18\frac{11}{20}$.

25. *To convert a Mixed Number into an Improper Fraction.*

Multiply the whole number by the denominator, and add the product to the numerator of the fraction.

$$6\frac{4}{5}=\frac{5\times 6+4}{5}=\frac{34}{5}:$$

for, by Art. 14, $6=\frac{6\times 5}{5}=\frac{30}{5}$,

and therefore, by Art. 24, $6+\frac{4}{5}=\frac{30+4}{5}=\frac{34}{5}$.

In the same manner,

$$16\frac{3}{4}=\frac{67}{4},\quad 78\frac{1}{2}=\frac{157}{2},\quad \text{and}\quad 92\frac{5}{63}=\frac{5801}{63}.$$

SUBTRACTION OF FRACTIONS.

26. (*a*) The difference between any two fractions, which have a common denominator, is found by taking the difference of their numerators, and subjoining the common denominator.

For the difference of the fractions is the difference between the number of the equal parts of the unit in each, and is, therefore, the difference of the numerators with the common denominator subjoined.

Ex. 1. $$\frac{5}{9}-\frac{4}{9}=\frac{5-4}{9}=\frac{1}{9}.$$

For the unit is divided into nine equal parts, and from five of those parts four are taken; there remains, therefore, one of those parts, or $\frac{1}{9}$.

(*b*) If the fractions have not a common denominator, transform them to others of equal value, with a common denominator, and proceed as before.

Ex. 2. Find the excess of $\frac{7}{12}$ above $\frac{3}{7}$.

$$\frac{7}{12}-\frac{3}{7}=\frac{49-36}{84}=\frac{13}{84}.$$

Ex. 3. From $34\frac{2}{5}$ subtract $6\frac{3}{7}$.

$$34\frac{2}{5}-6\frac{3}{7}=\frac{172}{5}-\frac{45}{7}=\frac{1204-225}{35}=\frac{979}{35}=27\frac{34}{35}:$$

or thus,

$$34\frac{2}{5}-6\frac{3}{7}=33\frac{7}{5}-6\frac{3}{7}=27+\frac{49-15}{35}=27\frac{34}{35}.$$

Ex. 4. Required the excess of $\frac{7}{11}$ above $\frac{3}{13}$. *Ans.* $\frac{58}{143}$.

Ex. 5. Required the excess of $33\frac{4}{7}$ above $21\frac{4}{9}$.

Ans. $12\frac{8}{63}$.

Ex. 6. Required the excess of $497\frac{13}{72}$ above $164\frac{5}{24}$.

Ans. $332\frac{35}{36}$.

Ex. 7. Required the excess of $3629\frac{13}{35}$ above $\frac{7}{15}$.

Ans. $3628\frac{19}{21}$.

27. When the operations of Addition and Subtraction are both implied, subtract the sum of those quantities which have the sign $-$ prefixed to them, from the sum of those which have $+$.

Ex. 1.

$$\frac{1}{3}+\frac{5}{7}-\frac{4}{9}+\frac{2}{5}-\frac{5}{12}=\left(\frac{1}{3}+\frac{5}{7}+\frac{2}{5}\right)-\left(\frac{4}{9}+\frac{5}{12}\right)$$
$$=\frac{152}{105}-\frac{31}{36}=\frac{789}{1260}.$$

Ex. 2. $\frac{5}{6}-\frac{3}{4}+3\frac{2}{3}-1\frac{1}{2}=2\frac{1}{4}$.

Ex. 3. $1-\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}=2\frac{43}{60}$.

Ex. 4. $2-\frac{1}{7}+\frac{12}{13}=2\frac{71}{91}$.

Ex. 5. $38\frac{1}{3}+\frac{1}{5}+24\frac{6}{7}-13\frac{4}{15}=50\frac{13}{105}$.

Ex. 6. $19\frac{3}{8}-\frac{2}{5}-3\frac{3}{11}+\frac{4}{7}=16\frac{843}{3080}$.

Ex. 7. $\frac{15}{16} - 5\frac{14}{15} + 7\frac{13}{14} - \frac{11}{12} = 2\frac{9}{560}$.

Ex. 8. $3 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{5}{6} - 2\frac{7}{8} + \frac{11}{12} = 1\frac{7}{24}$.

Ex. 9. $3\frac{3}{4} - 5\frac{4}{7} + 1\frac{63}{72} = \frac{3}{56}$.

Ex. 10. $57\frac{4}{17} + 3\frac{7}{50} + \frac{21}{850} - 35\frac{11}{90} = 25\frac{5}{18}$.

28. If the numerator of one fraction be added to the numerator of another, and the denominator of the one to the denominator of the other, the resulting fraction lies between the two.

$$\frac{1+1}{4+2} \text{ or } \frac{1}{3} \text{ lies between } \frac{1}{4} \text{ and } \frac{1}{2},$$

$$\frac{2+3}{3+5} \text{ or } \frac{5}{8} \text{ lies between } \frac{2}{3} \text{ and } \frac{3}{5}.$$

29. Adding the same number to both numerator and denominator of a fraction, brings the value of the fraction nearer to 1; subtracting the same number from both numerator and denominator, removes it farther from 1; *i.e.* the addition of the same number to both diminishes fractions greater than 1, and augments the value of fractions less than 1; the subtraction of the same number from both augments the value of fractions greater than 1, and diminishes fractions less than 1.

$\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}$, &c. is a continually decreasing series,

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$, &c. is a continually increasing series,

and *vice versâ,*

$\frac{8}{7}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}$ is an increasing series.

$\frac{7}{8}, \frac{6}{7}, \frac{5}{6}, \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}$ is a decreasing series,

MULTIPLICATION OF FRACTIONS.

30. To multiply one fraction by another, is to take such part or parts of the one as the other expresses.

To do this, multiply either fraction by the numerator, and divide it by the denominator of the other. (Arts. 11 and 12.)

The product thence arising may be multiplied by another fraction, and the process thus extended to any number of fractions.

Ex. 1. Multiply $\frac{2}{3}$ by $\frac{5}{9}$.

$$\frac{2}{3} \times \frac{5}{9} = \frac{2 \times 5}{3 \times 9} = \frac{10}{27}.$$

For $\frac{2}{3} \times \frac{5}{9} = \frac{5}{9}$ of $\frac{2}{3}$, and $\frac{1}{9}$ of $\frac{2}{3} = \frac{2}{27}$. (Art. 12.)

$$\therefore \frac{5}{9} \text{ of } \frac{2}{3} = \frac{2}{27} \times 5 = \frac{10}{27}. \text{ (Art. 11.)}$$

A fraction of a fraction, as $\frac{5}{9}$ of $\frac{2}{3}$, is called a *Compound Fraction.*

Ex. 2. Multiply $\frac{3}{8}$ by $\frac{4}{7}$.

$$\frac{3}{8} \times \frac{4}{7} = \frac{3}{2 \times 7} \text{ (Art. 11)} = \frac{3}{14}.$$

Ex. 3. Multiply $\frac{10}{11}$ by $\frac{3}{5}$.

$$\frac{10}{11} \times \frac{3}{5} = \frac{2 \times 3}{11} \text{ (Art. 12)} = \frac{6}{11}.$$

Ex. 4. Multiply $\frac{3}{4}$ by $\frac{6}{7}$, and the product thence arising by $\frac{7}{12}$.

$$\frac{3}{4} \times \frac{6}{7} = \frac{3 \times 6}{4 \times 7} = \frac{3 \times 3}{2 \times 7} \text{ (Art. 13)} = \frac{9}{14},$$

$$\text{and } \frac{9}{14} \times \frac{7}{12} = \frac{3}{2 \times 4} = \frac{3}{8}.$$

If the continued product of $\frac{3}{4}$, $\frac{6}{7}$, and $\frac{7}{12}$ were required, the operation would be performed thus:

$$\frac{3}{4}\times\frac{6}{7}\times\frac{7}{12}=\frac{3\times6\times7}{4\times7\times12}=\frac{3}{4\times2}\text{ (Art. 13 and Cor.)}=\frac{3}{8}.$$

Hence, the following

General Rule. Multiply all the numerators together for a new numerator, and the denominators for a new denominator.

Obs. In cancelling factors which are common to both the numerator and denominator, it is usual to make a mark through the factors reduced or cancelled: substituting an asterisk for the mark (for the convenience of printing) the process of finding the continued product, in Ex. 4, will stand thus:—

$$\frac{3}{4}\times\frac{6}{7}\times\frac{7}{12}=\frac{3\times{*6}\times{*7}}{4\times{*7}\times\underset{2}{{*12}}}=\frac{3}{8}.$$

When mixed numbers are given, they must be reduced to improper fractions.

Ex. 5. Required the continued product of $6\frac{3}{10}$, $7\frac{2}{9}$, and $43\frac{7}{13}$.

$$6\frac{3}{10}\times7\frac{2}{9}\times43\frac{7}{13}=\frac{\overset{7}{{*63}}\times\overset{*13}{{*65}}\times\overset{283}{{*566}}}{\underset{*2}{{*10}}\times{*9}\times{*13}}=1981.$$

Ex. 6. Find the value of $\frac{4}{7}$ of $\frac{9}{13}$ of $\frac{11}{21}$.

$$\frac{4}{7}\text{ of }\frac{9}{13}\text{ of }\frac{11}{21}=\frac{4\times\overset{3}{{*9}}\times11}{7\times13\times\underset{7}{{*21}}}=\frac{132}{637}.$$

Ex. 7. Find the value of $753\frac{5}{8} \times 12 \times \frac{1}{3}$.

$$753\frac{5}{8} \times 12 \times \frac{1}{3} = \frac{6029 \times \overset{4}{*12}}{\underset{2}{*8} \times *3} = \frac{6029}{2} = 3014\frac{1}{2}.$$

Ex. 8. Find the value of $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{7}$. *Ans.* $\frac{15}{56}$.

Ex. 9. Find the product of $\frac{3}{4}, \frac{6}{7}, \frac{4}{15}, \frac{11}{18}$ and $\frac{21}{23}$.

Ans. $\frac{11}{115}$.

Ex. 10. Find the value of.

$\frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8} \times \frac{8}{9} \times \frac{9}{10}$. *Ans.* $\frac{1}{5}$.

Ex. 11. Find the product of $2\frac{1}{3}$ and $3\frac{2}{5}$. *Ans.* $7\frac{14}{15}$.

Ex. 12. Find the value of $21\frac{5}{33} \times 11$. *Ans.* $232\frac{2}{3}$.

Ex. 13. Find the value of $\frac{3}{8} \times 2\frac{1}{16} \times 1\frac{1}{63} \times 3\frac{5}{87}$.

Ans. $2\frac{35}{87}$.

Ex. 14. Find the value of $\frac{12}{13} \times 2\frac{2}{3} \times 1\frac{1}{25} \times 1\frac{11}{64}$.

Ans. 3.

Ex. 15. Find the value of $12 \times 7\frac{5}{7} \times \frac{40}{63} \times \frac{49}{64} \times 1\frac{5}{72}$.

Ans. $48\frac{1}{8}$.

Ex. 16. Multiply $1\frac{1}{2} + \frac{2}{7}$ by $1\frac{14}{45} - \frac{5}{18}$,

$$\left(1\frac{1}{2} + \frac{2}{7}\right) \times \left(1\frac{14}{45} - \frac{5}{18}\right) = \frac{21+4}{14} \times \frac{118-25}{90}$$

$$= \frac{\overset{5}{\cancel{25}} \times \overset{31}{\cancel{93}}}{14 \times \underset{\underset{6}{\cancel{18}}}{\cancel{90}}} = \frac{155}{84} = 1\frac{71}{84}.$$

Ex. 17. Multiply $\frac{3}{4} + 3\frac{1}{2}$ by $\frac{3}{5} - \frac{1}{3}$. *Ans.* $1\frac{2}{15}$.

Ex. 18. Multiply $5\frac{1}{3} - 1\frac{5}{22}$ by $4\frac{2}{3} + 6\frac{7}{9}$.

Ans. $46\frac{589}{594}$.

DIVISION OF FRACTIONS.

31. To divide by a fraction, or to determine how often a fraction is contained in any quantity, invert the terms of the divisor, and proceed as in multiplication.

Ex. 1. Divide $\frac{3}{5}$ by $\frac{2}{7}$.

$$\frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \times \frac{7}{2} = \frac{21}{10} = 2\frac{1}{10}.$$

For $\frac{3}{5} \div 2 = \frac{3}{5 \times 2}$ (Art. 9); also $\frac{2}{7} = \frac{1}{7}$ of 2,

and is, therefore, contained seven times as often in $\frac{3}{5}$ as 2 is,

or $\left(\frac{3}{5 \times 2} \times 7\right)$ times;

whence $\frac{3}{5} \div \frac{2}{7} = \frac{3}{5 \times 2} \times 7 = \frac{3}{5} \times \frac{7}{2} =$ &c.

Ex. 2. Divide 6347 by $2\frac{3}{4}$.

$$6347 \div 2\frac{3}{4} = \overset{577}{{}^*6347} \times \frac{4}{{}^*11} \text{ (Art. 30, Obs.)} = 2308.$$

Ex. 3. Divide $\frac{18}{217}$ by $1\frac{2}{7}$.

$$\frac{18}{217} \div 1\frac{2}{7} = \frac{\overset{2}{{}^*18}}{\underset{31}{{}^*217}} \times \frac{{}^*7}{{}^*9} = \frac{2}{31}.$$

Ex. 4. Find the value of $4\frac{2}{3} \times 6\frac{3}{7} \div 7\frac{5}{9} \times \frac{6}{13} \div 1\frac{3}{14}$.

$$4\frac{2}{3} \times 6\frac{3}{7} \div 7\frac{5}{9} \times \frac{6}{13} \div 1\frac{3}{14} =$$
$$\frac{14}{3} \times \frac{45}{7} \times \frac{9}{68} \times \frac{6}{13} \times \frac{14}{17} = \frac{5670}{3757} = 1\frac{1913}{3757}.$$

Ex. 5. Divide $\frac{7}{8}$ by $\frac{4}{13}$. *Ans.* $2\frac{27}{32}$.

Ex. 6. Divide $3\frac{1}{13}$ by $19\frac{2}{43}$. *Ans.* $\frac{1720}{10647}$.

Ex. 7. Divide $\frac{2}{3}$ of 5 by $1\frac{1}{9}$. *Ans.* 3.

Ex. 8. Find the value of $3\frac{6}{7} \div 1\frac{1}{2} \times \frac{7}{9} \times 7\frac{1}{3} \div \frac{2}{15}$.

Ans. 110.

Ex. 9. Divide $5\frac{2}{9} + 2\frac{1}{3}$ by $3\frac{4}{7} - 1\frac{1}{2}$.

$$\left(5\frac{2}{9} + 2\frac{1}{3}\right) \div \left(3\frac{4}{7} - 1\frac{1}{2}\right) = \frac{47+21}{9} \div \frac{50-21}{14}$$
$$= \frac{68}{9} \times \frac{14}{29} = \frac{952}{261} = 3\frac{169}{261}.$$

Ex. 10. Divide $13 + \frac{2}{3}$ of 7 by $3\frac{2}{7} + \frac{1}{2}$ of $\frac{2}{3}$. *Ans.* $4\frac{67}{76}$.

When Division of Fractions is indicated by writing the dividend over the divisor, the fractional expressions thus formed are called *Complex Fractions.*

Ex. 11.

$$\frac{\frac{2}{3} \text{ of } \left(2\frac{1}{2} + \frac{1}{4}\right)}{\frac{3}{7} + \frac{1}{5} \text{ of } 9\frac{1}{2}} = \frac{\frac{2}{3} \times \frac{11}{4}}{\frac{3}{7} + \frac{19}{10}} = \frac{\frac{11}{6}}{\frac{30+133}{70}} = \frac{\overset{35}{11 \times {}^*70}}{\underset{3}{{}^*6 \times 163}} = \frac{385}{489}.$$

Ex. 12. Reduce $\dfrac{\frac{1}{4} \text{ of } \left(\frac{2}{5} + 2\frac{1}{3}\right) + 9}{137 + \frac{1}{6} \text{ of } 4\frac{1}{4}}$. *Ans.* $\frac{1162}{16525}$.

DECIMAL FRACTIONS.

32. As in the common system of notation of integers the local values of the figures increase from the right to the left in a ten-fold proportion, the first occupying the place of units, the second that of tens, the third that of hundreds, &c. so also when there are figures to the right of the units' place, they decrease in the same proportion from left to right, the first occupying the place of tenths, the second that of hundredths, the third that of thousandths, &c. and therefore, individually or collectively, represent fractions having ten or some product of tens for their denominators; whence they are called *decimal* fractions, to distinguish them from fractions written in the common form called *vulgar* fractions, and are separated from the integers by a point called the *decimal point.*

Thus 43.15 is equivalent to $40+3+\frac{1}{10}+\frac{5}{100}$ or $43\frac{15}{100}$.

Similarly, $.1 = \frac{1}{10}$, $.23 = \frac{23}{100}$, $.345 = \frac{345}{1000}$,

$$4.3 = 4\frac{3}{10},\quad 21.012 = 21\frac{012}{1000} \text{ or } 21\frac{12}{1000},$$

$$.001201 = \frac{1201}{1000000},\quad .00079 = \frac{79}{100000};$$

the denominators being 10 when there is only one figure to the right of the decimal point, 10×10 or 100 when there are two, $10 \times 10 \times 10$ or 1000 when there are three, and so on; the denominators of the corresponding vulgar fractions always containing as many ciphers as there are figures to the right of the decimal point, these figures being the numerators.

.1 is read *decimal one.*
.23 .... *decimal two, three.*
.345 .... *decimal three, four, five.*
21.012 .... *twenty-one, decimal nought, one, two.*

Cor. 1. Every cipher affixed to the left of a decimal diminishes its value ten-fold, by removing its digits one place more to the right of the point. Thus, .3, .03, .003, are respectively equal to $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$.

Cor. 2. If ciphers be annexed to the right of a decimal, its value is not altered. Thus, .3, .30, .300 being respectively equal to $\frac{3}{10}$, $\frac{30}{100}$, $\frac{300}{1000}$, are equal to one another.

33. Any number of decimals may be made to represent fractions having a common denominator, by annexing as many ciphers to the right as may be sufficient to render the number of decimal places the same in all.

Thus, .3000, .3300, .7045, are respectively equal to $\frac{3000}{10000}$, $\frac{3300}{10000}$, $\frac{7045}{10000}$.

34. *To express a vulgar fraction by means of a decimal fraction.*

If the given fraction be transformed into one having ten or some product of tens for its denominator, its decimal representative may be assigned by Art. 32.

Ex. 1. $$\frac{3}{40} = \frac{300}{4000} = \frac{300 \div 4}{4000 \div 4} = \frac{75}{1000} = .075.$$

Ex. 2. $$\frac{1}{37} = \frac{1000 \text{ \&c.}}{37000 \text{ \&c.}} = \frac{1000 \text{ \&c.} \div 37}{37000 \text{ \&c.} \div 37}$$
$$= \frac{27027 \text{ \&c.}}{1000000 \text{ \&c.}} = .\dot{0}2\dot{7}027 \text{ \&c.}$$

Ex. 3. $$\frac{17}{2250} = \frac{170000 \text{ \&c.}}{22500000 \text{ \&c.}} = \frac{170000 \text{ \&c.} \div 225}{22500000 \text{ \&c.} \div 225}$$
$$= \frac{755 \text{ \&c.}}{100000 \text{ \&c.}} = .00755 \text{ \&c.}$$

In Ex. 1. the exact value of the given fraction is expressed by means of a decimal, but this can be done only when the denominator is entirely composed of the factors two, or five, or both; for when it contains any other factor, as in Ex. 2 and 3, there exists no multiplier that will render the denominator of the required form, the division of the numerator by the denominator having no end; consequently only an approximation to the value of the fraction can be obtained: this approximation may, however, easily be made to differ but very little from the true value; for when the process is continued to only four places of decimals, the defect cannot exceed the ten-thousandth part of the unit (as will be seen by Art. 36); and the further the process is continued the nearer the approximation becomes, since by every succeeding step a decimal of inferior value is added.

When the division terminates, as in Ex. 1, the decimal is called a *terminating* decimal.

When the division does not terminate, the decimal always has some continually recurring digit or digits.

If each of the digits uniformly recur, as in Ex. 2, the decimal is called a *recurring* or *circulating* decimal.

If only part of them recur, as in Ex. 3, it is called a *mixed* recurring or circulating decimal.

A single recurring digit is sometimes distinguished by a point placed over it; thus $.007\dot{5}$; and a period by a point placed over its first and last digits, thus $.\dot{0}2\dot{7}$.

The transformations explained in this article may conveniently be effected as follows:

Rule. Divide the numerator of the proposed fraction by the denominator, annexing to the right of the former as many ciphers as may be required; and the quotient, with as many decimal places assigned to it as there are ciphers annexed, will be the decimal fraction required.

Ex. 4. Express $\frac{5}{8}$ by means of a decimal fraction.

$$\frac{5}{8} = \frac{5.000}{8} = .625.$$

Obs. In these examples, as in all others of the division of decimals, or mixed numbers containing decimals, by integers, it may be observed, that the first decimal figure in the quotient is the figure which results from the employment of the first decimal figure in the dividend; the second decimal in the quotient results from the employment of the second in the dividend, the third from the third, and so on.

Ex. 5. Express $\frac{17}{2250}$ by means of a decimal fraction.

$$\frac{17}{2550} = \frac{17.0000 \text{ \&c.}}{2250} = .0075.$$

Ex 6. Express $\frac{1}{4}$ and $\frac{8}{25}$ by decimal fractions.

Ans. .25 and .32.

Ex. 7. Express $\frac{1}{7}$ by a decimal fraction. *Ans.* $.\dot{1}4285\dot{7}$.

Ex. 8. Express $\frac{23}{55}$ by a decimal fraction. *Ans.* $.4\dot{1}\dot{8}$.

Ex. 9. Express $\frac{3}{16}$, $\frac{9}{500}$, $\frac{19}{1250}$, $\frac{170}{999}$, $\frac{32}{1665}$, and $47\frac{62}{333}$ in the form of decimals.

Ans. .1875, .018, .0152, $.\dot{1}7\dot{0}$, $.0\dot{1}9\dot{2}$, $47.\dot{1}8\dot{6}$.

35. ***To transform a decimal fraction to its equivalent vulgar fraction.***

(*a*) If the decimal be finite, write it over the product of as many tens as there are figures in the decimal (Art. 32), and reduce the fraction to its lowest terms.

Ex. 1. Transform .25 to its equivalent vulgar fraction.

$$.25 = \frac{25}{100} = \frac{1}{4}.$$

Ex. 2. Transform .5, .75, .625, .004, .4608, and .03725 to equivalent vulgar fractions.

Ans. $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{1}{250}$, $\frac{238}{625}$, $\frac{149}{4000}$.

(*b*) If the decimal consist entirely of recurring digits, write those digits over as many nines, and reduce the fraction as before.

Ex. 1. Transform $.\dot{3}$ to its equivalent vulgar fraction.

$$.\dot{3} = \frac{3}{9} = \frac{1}{3};$$

The truth of the process may be shewn thus:

The value of the required fraction = .333 &c.

$\therefore$ 10 times its value = 3.333 &c.

since, by Art. 32, each digit is multiplied by 10, by being removed one place to the left.

Whence, subtracting the former of these equals from the latter,

$$9 \text{ times its value} = 3;$$

$$\therefore \text{ the required fraction} = \frac{3}{9} = \frac{1}{3}.$$

Ex. 2. Transform $.\dot{2}\dot{7}$ to its equivalent vulgar fraction.

$$.\dot{2}\dot{7} = \frac{27}{99} = \frac{3}{11};$$

for the value of the required fraction $= .2727$ &c.

$$\therefore \ 100 \text{ times its value} = 27.2727 \text{ \&c.}$$

Whence, subtracting the former of these equals from the latter,

$$99 \text{ times its value} = 27;$$

$$\therefore \text{ the required fraction} = \frac{27}{99} = \frac{3}{11}.$$

Ex. 3. Transform $\dot{4}2.6\dot{3}$ to its equivalent, expressing the decimal by a vulgar fraction.

$$\dot{4}2.6\dot{3} = 42.\dot{6}34\dot{2} = 42\,\frac{6342}{9999} = 42\,\frac{2114}{3333}.$$

The truth of the process may be shewn, as in the two preceding examples, taking care to multiply by a sufficient number of tens to bring the recurring period to the left side of the decimal point, viz. $10 \times 10 \times 10 \times 10$ or 10000.

Ex. 4. Transform $.\dot{1}\dot{2}$, $.\dot{2}6\dot{5}$, $.\dot{2}3\dot{4}$, $.\dot{2}8571\dot{4}$, and $\dot{8}.3\dot{4}$ to their equivalents in vulgar fractions, and prove the accuracy of each result by reversing the transformation.

Ans. $\frac{4}{33}$, $\frac{265}{999}$, $\frac{78}{111}$, $\frac{2}{7}$, and $8\,\frac{116}{333}$.

(*c*) If the decimal contain both non-recurring and recurring digits, subtract the non-recurring digits from the

number which is composed of both the non-recurring and recurring digits, as integers; place the remainder over as many nines as there are recurring digits, followed by as many ciphers as there are non-recurring digits, and reduce the fraction.

Ex. 1. Transform $.41\dot{6}\dot{3}$ to its equivalent vulgar fraction.

$$.41\dot{6}\dot{3} = \frac{4163 - 41}{9900} = \frac{4122}{9900} = \frac{2061}{4950} = \frac{229}{550}.$$

The truth of the process may be shewn thus:

The value of the required fraction $= .416363 \ldots$

$\therefore$ 10000 times its value $= 4163.63 \ldots$

and 100 times its value $= 41.63 \ldots$

whence, taking the last of these equals from the second;

9900 times its value $= 4122$;

$\therefore$ the required fraction $= \dfrac{4122}{9900}$ as above.

Ex. 2. Transform $.02\dot{6}$ to its equivalent vulgar fraction

$$.02\dot{6} = \frac{26 - 2}{900} = \frac{24}{900} = \frac{2}{75}.$$

Ex. 3. Transform $.41\dot{3}$, $.5\dot{2}\dot{7}$, $.041\dot{6}$, $.012\dot{3}\dot{6}$, $.32\dot{5}7\dot{6}$ and $.372\dot{4}67$ to their equivalent vulgar fractions, and prove the correctness of each result by reversing the operation.

$$\textit{Ans.}\ \frac{31}{75},\ \frac{29}{55},\ \frac{1}{24},\ \frac{17}{1375},\ \frac{904}{2775},\ \frac{74419}{199800}.$$

In shewing the truth of the processes by which the transformations in this Art. are effected, subtraction of decimals is necessarily introduced, but in so obvious a manner as to be easily understood.

36. It is evident that the greatest possible decimal is .999 &c. to infinity.

$$.999 \text{ \&c. to infinity} = \frac{9}{9} = 1$$

$$.0999 \text{ \&c.} \ldots\ldots\ldots\ldots = \frac{9}{90} = \frac{1}{10}$$

$$.00999 \text{ \&c.} \ldots\ldots\ldots\ldots = \frac{9}{900} = \frac{1}{100}$$

&c.

And if any decimal .472013746 &c., for example, be given, any portion of it, say .472013, is less than the whole decimal, but .472014 is greater than the whole; the former being less than the proposed decimal by .000000746 &c. and the latter portion being greater than the former by $\frac{1}{1000000}$ or .000000999 &c.

Hence *an approximation to the value of a decimal, between which and the true value the difference shall not exceed any given difference of the form* $\frac{1}{1000\ldots}$, *may be obtained, by taking as many digits of the decimal as there are ciphers in the denominator of the given difference.*

It may be observed also that ·472014 is nearer to the value of the given decimal than .472013 is by .00000049 &c.; *and if, whenever the first rejected digit is* 5 *or greater than* 5, *there be added* 1 *to the last retained digit, the difference between the approximation and the true value will not exceed* ½ *of the given difference of the form* $\frac{1}{1000\ldots}$.

Thus, of the decimal .7182818,

in retaining one place only, write .7
.... two places .... .72
.... three .... .718
... . four .... .7183
.... five .... .71828
.... six .... .718282.

Ex. 1. Transform $\frac{17}{18}$ to a decimal fraction within $\frac{1}{1000}$ of the true value.

Here it will not be necessary to retain more than three places of decimals; annexing therefore three ciphers to the right of 17, and dividing by 18, (Art. 34.)

$$17.000 \div 18 = .944 \text{ within the given difference,}$$

$$\text{for } \frac{17}{18} - \frac{944}{1000} = \frac{8}{18000} = \frac{1}{2250},$$

which is less than half of the given difference, the first rejected digit of the decimal being less than 5.

Ex. 2. Transform $\frac{16}{17}$ to a decimal fraction approximating within $\frac{1}{2}$ of $\frac{1}{10000}$ or $\frac{1}{20000}$ of the true value.

$$16.0000 \div 17 = .94117\ldots = .9412 \text{ within the given difference,}$$

$$\text{for } \frac{9412}{10000} - \frac{16}{17} = \frac{4}{170000} = \frac{1}{42500}.$$

Ex. 3. Transform $\frac{64}{875}$ to a decimal fraction approximating within $\frac{1}{20000}$ of the true value. *Ans.* .0732.

ADDITION OF DECIMALS.

37. To find the sum of any number of decimals, or mixed numbers containing decimals, arrange the figures according to their local values, units under units, tens under tens, &c., also tenths under tenths, hundredths under hundredths, &c., then add them together as in integers, and place the decimal point in the sum under the points in the quantities proposed.

Ex. 1. Required the sum of .372, .4567, 14.3713, and 371.01.

$$\begin{array}{r} .372 \\ .4567 \\ 14.3713 \\ 371.01 \\ \hline 386.21 \end{array}$$

The operation here proceeds on the same principle as in addition of integers, for the local values of the figures throughout both the decimal and integral parts increase from the right to the left in the same proportion.

That it also is equivalent to the addition of fractions having a common denominator, may be seen by supplying ciphers according to Art. 33, and by the corresponding vulgar fractions, thus:

$$\begin{array}{r} .3720 \\ .4567 \\ 14.3713 \\ 371.0100 \\ \hline 386.2100 \end{array}$$

$$\text{For,}\quad .372 = \frac{3720}{10000},\quad .4567 = \frac{4567}{10000},$$

$$14.3713 = \frac{143713}{10000},\quad 371.01 = \frac{3710100}{10000},\quad \text{and}$$

$$\frac{3720 + 4567 + 143713 + 3710100}{10000} = \frac{3862100}{10000} = \frac{38621}{100}$$

$$= 386.21.$$

Ex. 2. Reduce the following series to the form of a decimal, accurate to 7 places:

$$1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \frac{1}{1 \times 2 \times 3 \times 4 \times 5}$$
$$+ \&c.$$

A point placed between the factors is frequently used as the sign of multiplication, and it will be convenient in this example, without being confounded with the decimal point.

$$1+\frac{1}{1}=2$$

$$\frac{1}{1.2}=\frac{1}{2}=.5.$$

$$\frac{1}{1.2.3}=\frac{1}{3}\text{ of }\frac{1}{2}=.16666666\ldots$$

$$\frac{1}{1.2.3.4}=\frac{1}{4}\text{ of }\frac{1}{1.2.3}=.04166666\ldots$$

$$\frac{1}{1.2.3.4.5}=\frac{1}{5}\text{ of }\frac{1}{1.2.3.4}=.00833333\ldots$$

$$\frac{1}{1.2.3.4.5.6}=\frac{1}{6}\text{ of }\frac{1}{1.2.3.4.5}=.00138888\ldots$$

$$\frac{1}{1.2.3.4.5.6.7}=\frac{1}{7}\text{ of }\frac{1}{1.2.3.4.5.6}=.00019846\ldots$$

$$\frac{1}{1.2.3.4.5.6.7.8}=\frac{1}{8}\text{ of }\frac{1}{1.2.3.4.5.6.7}=.00002480\ldots$$

$$\frac{1}{1.2.3.4.5.6.7.8.9}=\frac{1}{9}\text{ of }\frac{1}{1.2.3.4.5.6.7.8}=.00000275\ldots$$

$$\frac{1}{1.2.3.4.5.6.7.8.9.10}=\frac{1}{10}\text{ of }\frac{1}{1.2.3.4.5.6.7.8.9}=.00000027\ldots$$

Ans. 2.7182818.....

This series was selected by Lord Napier, the inventor of *Logarithms*, for the *Base* of the *Napierian System.*

Ex. 3. Required the sum of 346.1201, 24.00076, .004, 30.9. *Ans.* 401.02486.

Ex. 4. Required the sum of 232.15, .721, 36.999, 730.45797, .00203. *Ans.* 1000.33.

Ex. 5. Reduce $\frac{3}{4}+\frac{7}{8}+\frac{4}{25}+\frac{13}{128}+\frac{17}{1250}$ to the form of a decimal. *Ans.* 1.9001625.

Ex. 6. Reduce

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}+\frac{1}{2187}$$

to the form of a decimal accurate to five places.

Ans. 1.49977...

SUBTRACTION OF DECIMALS.

38. To subtract decimals, or mixed numbers containing them, write them under the quantities from which they are to be taken, according to the local values of the figures, suppose the requisite number of ciphers annexed to the right of either, to make the two contain the same number of decimal places; then subtract, as in integers, and place the decimal point in the result under the other points.

Ex. 1. From 37.001 subtract 4.41967.

$$\begin{array}{r} 37.001 \\ 4.41967 \\ \hline 32.58133 \end{array}$$

The nature of this process, like that of Addition, is obviously the same as in integers; its accuracy also may be shewn by means of the corresponding vulgar fractions.

Thus, $37.001=\frac{3700100}{100000}$, $4.41967=\frac{441967}{100000}$,

and $\frac{3700100-441967}{100000}=\frac{3258133}{100000}=32.58133$.

Ex. 2. Reduce $\frac{3}{8} - \frac{1}{25} - \frac{13}{625} + \frac{9}{400} - \frac{3}{10}$ to a decimal fraction.

$$\frac{3}{8} = .375 \qquad \frac{1}{25} = .04$$

$$\frac{9}{400} = .0225 \qquad \frac{13}{625} = .0208$$

$$\frac{3}{10} = .3$$

From	.3975	.3608
Subtract	.3608	
Ans.	.0367	

Ex. 3. From 36.0017 subtract 35.10046. *Ans.* .90124.

Ex. 4. From .90124 subtract .017923. *Ans.* .883317.

Ex. 5. Reduce $4\frac{3}{4} - \frac{1}{5} + \frac{13}{128} - \frac{53}{125} - \frac{17}{1250}$ to the form of a decimal. *Ans.* 4.2139625.

MULTIPLICATION OF DECIMALS.

39. To find the product of two decimals, or mixed numbers containing decimals, multiply as in integers, and point off as many decimal places in the product as there are in both factors.

Ex. 1. Find the product of 31.43 and .3,

$$31.43 \times .3 = 9.429.$$

The corresponding process by vulgar fractions being

$$31.43 \times .3 = \frac{3143}{100} \times \frac{3}{10} = \frac{9429}{1000} = 9.429.$$

Ex. 2. Multiply 34.5 by 2.73.

$$\begin{array}{r} 34.5 \\ 2.73 \\ \hline 1035 \\ 2415 \\ 690 \\ \hline 94.185 \end{array}$$

The corresponding process by vulgar fractions being

$$34.5 \times 2.73 = \frac{345}{10} \times \frac{273}{100} = \frac{94185}{1000} = 94.185.$$

The multiplication of a decimal by a number of the form 1000... is performed by removing the decimal point as many places to the right as there are ciphers in the multiplier; for the local values of the figures increase ten-fold for every place the point is removed to the right. (Art. 32.)

Ex. 3. $3.24 \times 10 = 32.4$, $.279 \times 10000 = 2790$.

When there are not so many figures in the product as decimal places in both factors, the defect must be supplied by annexing ciphers to the left.

Ex. 4. Required the product of .285 and .302.

$$.285 \times .302 = .08607,$$

$$\text{for } .285 \times .302 = \frac{285}{1000} \times \frac{302}{1000} = \frac{86070}{1000000} = .08607.$$

The operation being performed thus,

$$\begin{array}{r} .285 \\ .302 \\ \hline 570 \\ 855 \\ \hline .086070 \end{array}$$

Ex. 5. Required the product of 3706.205 and 34.005.

Ans. 126029.501025.

Ex. 6. Required the product of 4.26 and 5000.

Ans. 21300.

Ex. 7. Required the product of 625 and .0208.

Ans. 13.

Ex. 8. Required the product of .326 and .004.

Ans. .001304.

Ex 9. Required the product of 3.791 and .0036.

Ans. .0136476.

DIVISION OF DECIMALS.

40. Division of decimals is performed as in integers, observing to point off as many decimal places in the quotient as the number of decimal places in the dividend exceeds the number in the divisor; for, by the nature of division, the product of the divisor and quotient is equal to the dividend, and, by last Article, the number of decimal places in the product is equal to the number in both factors.

Ex. 1. Divide 617.42070012 by 13.0459.

```
13.0459) 617.42070012 (47.3268
         521836
         ------
          955847
          913213
          ------
           426340
           391377
           ------
            349630
            260918
            ------
             887121
             782754
             -------
             1043672
             1043672
             -------
              . . . .
```

When the quotient does not contain the requisite number of figures, the deficiency must be supplied by annexing ciphers to the left.

Ex. 2. Divide .02961 by .7.

$$.02961 \div .7 = .0423, \text{ for}$$

$$.02961 \div .7 = \frac{2961}{100000} \div \frac{7}{10} = \frac{2961}{100000} \times \frac{10}{7} = \frac{423}{10000} = .0423.$$

Ex. 3. Divide 3.6494 by 142.

$$3.6494 \div 142 = .0257, \text{ (Art. 34. Obs.)}$$

$$\text{for } 3.6494 \div 142 = \frac{36494}{10000} \div 142 = \frac{257}{10000} = .0257.$$

If the divisor be an integer of the form 1000..., remove the decimal point as many places to the left as there are ciphers in the divisor.

Ex. 4. $78.5 \div 10 = 7.85$, $57.46 \div 1000 = .05746$.

When the dividend has not so many decimal places as the divisor, annex as many ciphers to the right of the former as will supply the defect, and more if required.

Whenever there is a remainder, a cipher may be annexed to the right of the dividend, and the division continued.

Ex. 5. Divide 14.4 by .12, and 3 by 7.5.

```
.12) 14.40          7.5) 3.00 (.4
     -----               300
      120                ---
                          ..
```

The necessity for annexing the ciphers to the right of these dividends, in order to obtain the true results, will be seen by the corresponding vulgar fractions.

$$14.4 \div .12 = \frac{144}{10} \div \frac{12}{100} = \frac{144}{10} \times \frac{100}{12} = 12 \times 10 = 120.$$

$$3 \div 7.5 = 3 \div \frac{75}{10} = 3 \times \frac{10}{75} = \frac{10}{25} = \frac{40}{100} = .4$$

Ex. 6. Divide 65 by .342.

```
.342) 65.000 (190.0584...
      342
      ----
      3080
      3078
      --------
         2000
         1710
         -----
          2900
          2736
          -----
           1640
           1368
           -----
```

$$\frac{272}{342}=\frac{136}{171}.$$

The value of the remainder is

$$\frac{136}{171} \text{ of } \frac{1}{10000}=\frac{136}{1710000}=\frac{17}{213750}.$$

By vulgar fractions

$$65 \div .342 = 65 \div \frac{342}{1000} = 65 \times \frac{1000}{342} = \frac{65000}{342} = 190.0584...$$

It may be observed, that the first figure in the quotient always occupies the same position with regard to the decimal point, as that figure of the dividend which stands immediately over the units' figure in the first product, ciphers being supposed to be annexed to the left of the integers, or to the right of the decimals, when required, in order to obtain that figure. Thus, in Ex. 1, the units' figure of the first product is the 2, and the figure 1 of the dividend, which stands immediately over it, is in the place of tens, therefore the first figure of the quotient should occupy the place of tens.

Hence the place of the decimal point may be determined at the beginning of the operation, and at the conclusion of

the operation, the accuracy of its position, so determined, may be tested by comparing the number of decimal places in the divisor and quotient together, with the number of decimal places in the dividend.

Ex. 7. Divide 94.185 by 2.73, and prove the truth of the result by multiplying the quotient by the divisor.

Ans. 34.5.

Ex. 8. Divide 14.4 by 1.2, 1.44 by .12, and .144 by .012; and prove the truth of the result in each case, both by reversing the operation and by the corresponding vulgar fractions.

Ex. 9. Divide 5.3721 by .9, and prove the truth of the result. *Ans.* 5.969.

Ex. 10. Divide .0136476 by 3.791, and prove the truth of the result. *Ans.* .0036.

Ex. 11. Divide 4.567323 by 300, and prove the truth of the result. *Ans.* .01522441.

Ex. 12. Divide .000208 by .016, and prove the truth of the result. *Ans.* .013.

Ex. 13. Divide 134.39796 by .0316, and prove the truth of the result. *Ans.* 42.531.

Ex. 14. Divide 125 by .025, and prove the truth of the result. *Ans.* 500.

Ex. 15. Find the quotient of 37.24 by 2.9 to six places of decimals. *Ans.* 12.841379 &c.

Ex. 16. Divide 22 by 166.5, and prove the truth of the result by the corresponding vulgar fractions. *Ans.* $.\dot{1}3\dot{2}$.

Ex. 17. Divide 1.7 by 1375, mark the recurring period, and prove the truth of the result by the corresponding vulgar fractions. *Ans.* $.012\dot{3}\dot{6}$.

Ex. 18. Find the quotient of 904 by 2775, mark the recurring period, and prove the truth of the result by the corresponding vulgar fractions. *Ans.* $.32\dot{5}7\dot{6}$.

41. The fundamental operations of Arithmetic may generally be performed upon recurring decimals by the foregoing rules, with a sufficient approach to accuracy, if only a few decimal digits be retained (Art. 38.); but, if required, they may, in any case, be performed accurately by means of their equivalent vulgar fractions.

Ex. 1. Let the sum, difference, product, and quotient of $.\dot{4}$ and $.2\dot{1}\dot{7}$ be required.

$$.\dot{4} = \frac{4}{9}, \quad \text{and } .2\dot{1}\dot{7} = \frac{215}{990} = \frac{43}{198};$$

$$\text{hence the sum} = \frac{4}{9} + \frac{43}{198} = \frac{131}{198} = .6\dot{6}\dot{1},$$

$$\text{the difference} = \frac{4}{9} - \frac{43}{198} = \frac{45}{198} = .2\dot{2}\dot{7},$$

$$\text{the product} = \frac{4}{9} \times \frac{43}{198} = \frac{86}{891} = .09652\ \&c.$$

$$\text{and the quotient} = \frac{4}{9} \div \frac{43}{198} = \frac{88}{43} = 2.04651\ \&c.$$

COMPOUND NUMBERS.

42. In this kingdom the different quantities of the same kind are represented by numbers under different denominations, connected, not by a common multiplier, but irregularly; and those expressions which contain numbers under different denominations are called *Compound Numbers*.

These various denominations, and their relations to each other, are exhibited in the following Tables.

I. TABLE OF MONEY.

4 farthings (*q*)	are	1 penny	..	*d.*
12*d.*	..	1 shilling	..	*s.*
20*s.*	..	1 pound	..	£.

			£.	*s.*	*d.*
A groat ..	is	..	0	0	4
A tester ..	..	..	0	0	6
A crown ..	..	..	0	5	0
A noble ..	..	..	0	6	8
An angel, or half-sovereign			0	10	0
A mark ..	..	..	0	13	4
A sovereign	..	..	1	0	0
A guinea ..	..	..	1	1	0
A Carolus	..	..	1	3	0
A Jacobus	..	..	1	5	0
A moidore	..	..	1	7	0
A six-and-thirty ..		..	1	16	0

II. WEIGHT.

Avoirdupois Weight.

16 drams	are	..	1 ounce	..	*oz.*
16 oz. ..	..	..	1 pound	..	*lb.*
14 lbs. ..	..	..	1 stone	..	*st.*
2 stone or 28 lbs.		..	1 quarter	..	*qr.*
4 qrs. or 112 lbs.		..	1 hundredweight	..	*cwt.*
20 cwt. ..	..	..	1 ton		

Troy Weight.

24 grains	are	1 pennyweight	..	*dwt.*
20 dwt.	..	1 oz.		
12 oz.	..	1 lb.		

Apothecaries' Weight.

20 grains	..	are	..	1 scruple	..	℈.
3 sc.	..	..	..	1 dram	..	ʒ.
8 drs.	..	..	..	1 oz.	..	℥.
12 oz.	..	..	..	1 lb.	..	℔.

Wool Weight.

7 lbs.	..	are	..	1 clove
2 cloves or 14 lbs.		..	..	1 stone
2 st.	..	..	..	1 tod
6½ tods ..	..	..	..	1 wey
2 weys ..	..	..	..	1 sack
12 sacks ..	..	..	..	1 last
240 lbs. ..	..	..	..	1 pack

III. LINEAR MEASURE.

12 inches	..	are	..	1 foot
3 feet ..	..	..	..	1 yard
5½ yds. ..	..	..	..	1 pole
40 pls. or 220 yds.		..	..	1 furlong
8 fur. or 1760 yds.		..	..	1 mile
3 mls. ..	..	..	..	1 league

A line ..	..	is	..	1-12th of an inch
A barleycorn	..	..	..	1-3rd of an inch
A palm	..	..	..	3 in.
A hand	..	..	..	4 in.
A span ..	..	..	..	9 in.
A cubit	..	..	..	18 in.
A pace ..	..	..	..	5 feet
A fathom	..	..	..	6 ft.

A degree is 60 geographical miles, or a little more than 69 statute miles.

Cloth Measure.

2¼ in. ..	..	are	..	1 nail
4 nls. ..	..	..	..	1 quarter
4 qrs. ..	..	..	..	1 yard
3 qrs. ..	..	..	..	1 Flemish ell (nearly)
5 qrs. ..	..	..	..	1 English ell
6 qrs. ..	..	..	..	1 French ell (nearly)

IV. SUPERFICIAL MEASURE.

144 square in.	are	1 sq. ft.
9 sq. ft. ..	..	1 sq. yd.
30¼ sq. yds.	..	1 sq. pl. or perch
40 sq. pls. ..	..	1 rood
4 roods ..	..	1 acre
640 acres ..	..	1 sq. mile

V. SOLID MEASURE.

1728 cubic in.	..	are ..	1 cubic ft.
27 cu. ft.	..	.. ..	1 cu. yd.
40 cu. ft.	..	.. ..	1 load of rough timber
50 cu. ft.	..	.. ..	1 load of squared timber
42 cu. ft.	..	.. ..	1 ton of shipping

VI. MEASURE OF CAPACITY.

4 gills	..	are ..	1 pint
2 pts.	..	.. ..	1 quart
4 quarts	..	.. ..	1 gallon

Ale and Beer Measure.

9 gallons	..	are ..	1 firkin
2 fir. or 18 gals.	..	..	1 kilderkin
2 kil. or 36 gals.	..	..	1 barrel
$1\frac{1}{2}$ bar. or 54 gals.	..	..	1 hogshead . *hhd.*
2 bar.	..	.. ..	1 puncheon
2 hhds.	..	.. ..	1 butt
2 butts	..	.. ..	1 tun

Wine & Spirit Measure.

10 gallons	..	are ..	1 anker
18 gals.	..	.. ..	1 runlet
42 gals.	..	.. ..	1 tierce
2 tierces or 84 gals.	..	..	1 puncheon
63 gals.	..	.. ..	1 hhd.
2 hhds. or 126 gals.	..	..	1 pipe
2 pipes or 252 gals.	..	..	1 tun

Dry Measure.

2 quarts	..	are ..	1 pottle
2 pottles	..	.. ..	1 gallon
2 gallons	..	.. ..	1 peck
4 pecks	..	.. ..	1 bushel
2 bush.	..	.. ..	1 strike
4 bush.	..	.. ..	1 comb
2 combs	..	.. ..	1 quarter
4 qrs.	..	.. ..	1 chaldron of corn
20 combs	..	.. ..	1 last
3 bush.	..	.. ..	1 sack of coals
36 bush.	..	.. ..	1 chaldron of coals

VII. MEASURE OF TIME.

60 seconds	..	are	..	1 minute
60 min.	..	..	..	1 hour
24 hrs.	..	..	..	1 day
7 days	..	..	..	1 week

VIII. CIRCULAR MEASURE.

60 seconds	..	are	..	1 minute
60 min.	..	..	..	1 degree
90 deg.	..	..	..	1 quadrant
4 quadrants	..	..	..	1 circumference

IX. NUMBER.

12 units	..	are	..	1 dozen
12 dozen	..	..	..	1 gross
144 dozen	..	..	..	a great gross
20 units	..	..	..	1 score
6 scores	..	..	..	a long hundred
24 sheets of paper		..	..	1 quire
20 quires	..	..	..	1 ream
2 reams	..	..	..	1 bundle
5 bundles	..	..	..	1 bale

Explanatory Notes.

43. TABLE OF MONEY.—The money expressed by the denominations in this table is called *Sterling* money, to distinguish it from Stock, &c., which is merely nominal.

The *Pound Sterling* is represented by a gold coin called a *Sovereign*, weighing 123.274, or about $123\frac{1}{4}$ grains of standard gold, which consists of 11-12ths of pure gold and 1-12th of alloy.

The *Shilling* is a silver coin, weighing about $87\frac{2}{11}$ grains of standard silver, which consists of 40 parts of pure silver to 3 of alloy.

The *Penny*, *Half-penny*, and *Farthing*, are copper coins, of which the penny weighs nearly an ounce.

The half-penny and farthings are generally expressed by the fractions of a penny; thus, $\frac{1}{4}d.$, $\frac{1}{2}d.$, $\frac{3}{4}d.$

Besides these there are now current the four-penny, six-penny, half-crown, and crown-piece in silver, and half-sovereign in gold: the other pieces enumerated in this table are not now in common use, but are frequently met with in the collections of antiquarians, and in old documents.

WEIGHT.—By 5 and 6 Wm. IV., c. 43, sec. 9, it was enacted,—That from and after the 1st of January, 1836, all coals, slack, culm, and cannel of every description, should be sold by weight and not by measure; and every person who should from and after the 1st of January, 1836, sell any coals, slack, culm, or cannel of any description, by measure and not by weight, should on conviction be liable to a penalty not exceeding 40*s.* for every such sale.

And by sec. 10, it was enacted,—That from and after the passing of this Act, all articles sold by weight should be sold by avoirdupois weight, except gold, silver, platina, diamonds, or other precious stones, which may be sold by troy weight, and drugs, which, when sold by retail, may be sold by apothecaries weight.

The ounce troy is greater than the ounce avoirdupois; but the pound troy is less then the pound avoirdupois: 175 oz. troy are equal to 192 oz. avoirdupois; but 175 lbs. troy are equal to only 144 lbs. avoirdupois.

The grain, ounce, and pound, are the same in apothecaries weight as in troy weight.

In wool weight the pound is the same as in avoirdupois weight.

A firkin of butter is 4 stone, or 56 lbs.; a fodder of lead is $19\frac{1}{2}$ cwt.; and articles of silk are sometimes weighed by a pound of 24 oz., called a *Great Pound.*

A sack of flour weighs 20 stone, and is about 5 imperial bushels.

LINEAR MEASURE.—By this measure are computed the linear dimensions of all magnitudes.

A barleycorn, or grain of barley, is supposed to have been the original element of linear measure, and a grain of wheat the element of weight.

SUPERFICIAL MEASURE.—A square inch, foot, &c., is the superficial magnitude of a square, of which each of the sides is a linear inch, foot, &c.; and a square number is produced by multiplying any number into itself; thus, the square number $144 = 12 \times 12$, $9 = 3 \times 3$, &c.

By superficial measure are computed all kinds of surfaces, as land, paving, plastering, &c.

Land is measured by means of a chain, consisting of 100 equal links, called Gunter's chain. Its length is 4 poles; therefore 10 square chains, or 100,000 square links, make an acre.

Glazing, mason's flat work, &c., are measured by the square foot; painting, paving, plastering, &c., by the square yard; flooring, roofing, &c., by the square of 100 feet.

Brick-work is measured by the rod of $16\frac{1}{2}$ feet; and $272\frac{1}{4}$ square feet of brick-work, a brick-and-a-half thick are reckoned a square rod.

SOLID MEASURE.—A cubic inch, foot, &c., is the solid content of a cube, or figure contained by six equal squares, of which each of the edges is a linear inch, foot, &c.; and a cubic number is the continued product of any number taken thrice; thus, the cubic number $1728 = 12 \times 12 \times 12$, $27 = 3 \times 3 \times 3$, &c.

The solid content of a parallelòpiped, or figure contained by six quadrilateral surfaces, whereof every opposite two are parallel, is the continued product of its length, breadth, and thickness.

Measure of Time.—The mean time between two passages of the Sun across the meridian of any place, is termed a *Mean Solar Day*, which is supposed to be equally divided into the parts mentioned in the table; and from astronomical observations and calculations it appears, that the time between the Sun's leaving a fixed point in his path, the ecliptic, and returning to it again, is 365.242264 such days, or 365 days, 5 hrs., 48 min., $51\frac{3}{5}$ sec. nearly, which is therefore called a *Solar Year.*

The principal regulators of our Calendar were Romulus, Numa, Julius Cæsar, and Pope Gregory.

Cæsar finding that, notwithstanding the successive efforts of Romulus and of Numa to adapt the civil to the solar year, the calendar was very imperfect, undertook, with the aid of some of the ablest astronomers of his time, to remedy its defects. These astronomers supposed the solar year to consist of 365 days, 6 hours, or $365\frac{1}{4}$ days; and to obviate the inconvenience which would arise from the civil year not consisting of an exact number of days, it was ordained that every fourth year should be reckoned to consist of 366 days, and the remaining three of 365 days: so that, on this supposition as to the length of the solar year, the Sun would return, at the end of every four years, to the place in the ecliptic which he left at their commencement. The calendar thus adjusted was termed the *Julian Calendar.*

The additional day for every fourth year was obtained by repeating the sixth of the calends of March in the Roman calendar, which corresponds with the 24th of February in ours; and the year in which it was inserted was therefore termed *Intercalary*, or *Bissextile.*

This regulation was afterwards so applied to the years of the Christian dispensation, that whenever the number of

years since the birth of Christ was exactly divisible by 4, that year consisted of 366 days, and was termed *Leap Year;* the month of February having 29 days in leap year, and only 28 days in each of the remaining three years.

But the solar year being 365.242264 days instead of 365$\frac{1}{4}$ or 365.25 days, there arose in one year an error of the difference of these two mixed numbers, viz. of .007736 of a day, which placed the end of the civil year .007736 of a day after the time of the Sun's return to the point in the ecliptic which he left at its commencement.

This error was continually increasing at the rate of .007736 of a day per annum; and by multiplying this decimal fraction by 400, it appears that in 400 years the error amounted to 3.0944 days, or 3 days, 2 hrs., 16 min., nearly.

In 1582, Pope Gregory, finding that the error had accumulated to 10 days, cut off ten days from the calendar, and promulgated the following correction of it.

That whereas by the Julian Calendar the last year of every century, (as 1600, 1700, 1800, 1900,) would be a leap year, only every fourth of such years should be a leap year; and that of these 1600 should be the first leap year: so that whenever the number indicating the century concluded, (as 16, 17, 18, or 19,) should be divisible by 4, the year should be a leap year, but not otherwise.

Thus, in 400 years, 3 leap years were changed for ordinary years, and 3 days of the calendar thereby dropped, leaving the accumulated error of 400 years only about 2 hours 16 min., which would not amount to a day in 4000 years. With this correction the calendar is called the *Gregorian Calendar*, or the *New Style*, the Julian Calendar being now termed the *Old Style*.

Hence the rule is, that every year, of which the date is

divisible by 4, is a leap year, unless that date express the last year of a century; in which case the year is a leap year only when the number indicating the century concluded is divisible by 4.

The Gregorian Calendar, or the new style, was adopted in England on the 2nd of September, 1752, when the error amounted to 11 days. Had the old style been continued, the error would now have been 12 days, because by it 1800 would have been a leap year, which by the new style it was not: and accordingly we have old Christmas-day, old Midsummer-day, &c. falling 12 days later than Christmas-day, Midsummer-day, &c., as fixed by the present system.

Of the twelve calendar months, viz. January, February, March, April, May, June, July, August, September, October, November, and December,—April, June, September, and November consist of 30 days; February has only 28 days, except in leap years, when it has 29; and all the rest have 31 days.

The word month is also used to denote 4 weeks, or 28 days, which is nearly a *Lunar Month*, and sometimes to signify a twelfth part of the year.

44. With a view to an invariable standard, and one which cannot be lost or destroyed, the linear inch has been referred to a pendulum vibrating seconds in a non-resisting medium at the level of the sea at Greenwich or London. The pendulum being divided into 391392 equal parts, the linear inch is defined to be 10000 of these parts; the length of the seconds' pendulum is therefore 39.1392 inches.

The imperial gallon is defined to be 277.274 cubic inches: and the pound avoirdupois the weight of one-tenth part of an imperial gallon of distilled water.

REDUCTION.

45. The operation by which the denominations of quantities are changed, whilst their values remain the same, is called *Reduction*.

Ex. 1. Reduce £31. 5*s*. $6\frac{1}{2}$*d*. to farthings.

Since there are 20*s*. in £1, in any number of pounds there are 20 times as many shillings, therefore the denomination of any number is changed from pounds to shillings by multiplying that number by 20; thus

$$£31 = 31 \times 20s. = 620s.,$$

and with the addition of the 5*s*. given, there are 625*s*. in £31. 5*s*. Also the number of pence in £31. 5*s*. 6*d*. is obtained by multiplying the 625*s*. by 12, and adding to the product the 6*d*. given; and by a similar process the number of farthings in £31. 5*s*. $6\frac{1}{2}$*d*. is found to be 30026.

The operation stands thus,

$$\begin{array}{r} £. \;\; s. \;\; d. \\ 31 \,.\, 5 \,.\, 6\tfrac{1}{2}. \\ 20 \\ \hline 625s. \\ 12 \\ \hline 7506d. \\ 4 \\ \hline 30026q. \\ \hline \end{array}$$

The mode of reasoning employed in this example being applicable to every example of *reduction to a lower denomination*, the rule is as follows.

RULE 1. Multiply the number of the highest denomination in the quantity proposed by that number of the next

inferior denomination which is equivalent to an unit of the highest denomination, and to the product add the number of the inferior denomination; and repeat the process for each succeeding denomination until the one required is obtained.

Ex. 2. Reduce 30026 farthings to pence, shillings, and pounds.

Since there are 4 farthings in 1 penny, there are in any number of farthings but one-fourth of the number of pence; therefore the denomination of any number is changed from farthings to pence by dividing that number by 4; thus

$$30026q. = \frac{30026}{4} d. = 7506\tfrac{1}{2}d.;$$

similarly

$$7506\tfrac{1}{2}d. = \frac{7506\frac{1}{2}}{12} s. = 625s.\ 6\tfrac{1}{2}d. = \frac{625}{20} £.\ 6\tfrac{1}{2}d. = £31.\ 5s.\ 6\tfrac{1}{2}d.$$

The operation is written thus,

$$\begin{array}{r l}
4) & \underline{30026q.} \\
12) & \underline{7506\frac{1}{2}d.} \\
20) & \underline{625.\ 6\frac{1}{2}d.} \\
 & £31.5.6\tfrac{1}{2}.
\end{array}$$

Hence, Rule 2. *To reduce quantities to their equivalents in higher denominations,* divide by the numbers which connect the different denominations in order, and to the quotients annex the remainders, assigning to each the denominations of the dividends from which they respectively arise.

The truth of the process may be confirmed, in each case, by reversing the operation.

Ex. 3. Required the number of pounds in 380 guineas.

Reducing the given sum to shillings by Rule 1, and then applying Rule 2, we have

$$\begin{array}{r l}
 & 380 \text{ guineas} \\
 & \quad 3 \times 7 = 21 \\
\hline
 & 1140 \\
 & \quad\; 7 \\
\hline
20) & 7980s. \\
\hline
 & 399£. \quad \textit{Ans.}
\end{array}$$

Ex. 4. Reduce $\frac{1}{48}$ of a yard to the fraction of an inch.

Multiplying by the numbers which connect the successive denominations, according to Rule 1, we have

$$\frac{1}{48}\,yd. = \frac{1 \times 3 \times 12}{48}\,in. = \frac{3}{4}\,in.$$

Ex. 5. What fraction of a pound is $\frac{2}{3}$ of a penny?

Dividing by the numbers which connect the denominations in order, by Rule 2, we have

$$\frac{2}{3}d. = £\frac{2}{3 \times 12 \times 20} = £\frac{1}{360}.$$

Ex. 6. What fraction of a guinea is $\frac{3}{8}$ of a pound?

$$£\frac{3}{8} = \frac{3 \times 20}{8}s. = \frac{15}{2 \times 21}\;gui. = \frac{5}{14} \text{ of a guinea.}$$

Ex. 7. What fraction of a pound is $10d.$?

$$10d. = £\frac{10}{12 \times 20} = £\frac{1}{24}.$$

Ex. 8. Reduce $10s.\ 5\frac{3}{4}d.$ to the fraction of a pound.

Reducing the given sum to the lowest given denomination by Rule 1, and then applying Rule 2, we have

$$10s.\ 5\tfrac{3}{4}d. = 503q. = £\frac{503}{4 \times 12 \times 20} = £\frac{503}{960}.$$

Ex. 9. What fraction of an acre is 1 rood 24 perches?

$$1 \text{ rood } 24 \text{ p.} = 64 \text{ p} = \frac{64}{40 \times 4} \text{ of an acre} = \frac{2}{5} \text{ of an acre.}$$

Ex. 10. What fraction of a guinea is *5s. 6d.*?

$$5s.\ 6d. = 66d. = \frac{66}{12 \times 21}\ gui. = \frac{11}{42} \text{ of a guinea.}$$

Or, expressing the given sum and the unit of the required denomination in their highest common denomination, thus

5s. 6d. = 11 sixpences, and a guinea = 42 sixpences;

$$\therefore\ 5s.\ 6d. = \frac{11}{42} \text{ of a guinea.}$$

Ex. 11. Find the value of $\frac{3}{7}$ of £1.

$$£\frac{3}{7} = \frac{3 \times 20}{7}s. = 8\frac{4}{7}s. = 8s.\ \frac{4 \times 12}{7}d. = 8s.\ 6\frac{6}{7}d.$$

$$= 8s.\ 6d.\ \frac{6 \times 4}{7}q. = 8s.\ 6d.\ 3\frac{3}{7}q.$$

The operation is usually written thus,

```
     20s.
      3
    ----
  7) 60
    ------
      8 ... 4
           12
          ----
        7) 48
          ------
           6 ... 6
                 4
                ---
             7) 24
                ------
                 3 ... 3
```

Ans. $8s.\ 6d.\ 3\frac{3}{7}q.$

Ex. 12. Find the value of $\frac{2}{5}$ of 4 lbs. troy.

$\frac{2}{5}$ of 4 lbs. $= \frac{8}{5}$ lb. (Art. 30.) $= 1\frac{3}{5}$ lb. $=$ 1 lb. $\frac{3 \times 12}{5}$ oz.

$=$ 1 lb. $7\frac{1}{5}$ oz. $=$ 1 lb. 7 oz. $\frac{1 \times 20}{5}$ dwt. $=$ 1 lb. 7 oz. 4 dwt.

Or thus,

$$\begin{array}{rl}
 & 4 \text{ lbs.} \\
 & \underline{2} \\
5) & \underline{8} \\
 & 1 \ldots 3 \\
 & \quad \underline{12} \\
5) & \quad \underline{36} \text{ oz.} \\
 & \quad 7 \ldots 1 \\
 & \qquad \underline{20} \\
5) & \qquad \underline{20} \text{ dwt.} \\
 & \qquad \underline{4}
\end{array}$$

Ans. 1 lb. 7 oz. 4 dwt.

Ex. 13. Find the value of $\frac{2}{7}$ of 3 cwt.

$$\begin{array}{rl}
 & 3 \text{ cwt.} \\
 & \underline{2} \\
7) & 6 \\
 & \underline{4} \\
7) & \underline{24} \text{ qrs.} \\
 & 3 \ldots 3 \\
 & \quad \underline{28} \\
7) & \quad \underline{84} \text{ lbs.} \\
 & \quad \underline{12}
\end{array}$$

Ans. 3 qrs. 12 lbs.

Ex. 14. Reduce .023076 cwt. to the denomination of pounds.

Multiplying by the numbers which connect the successive denominations, according to Rule 1, we have

$$\begin{array}{rl} .023076 & \text{cwt.} \\ 4 & \\ \hline .092304 & \text{qr.} \\ 28 & \\ \hline 2.584512 & \text{lbs.} \end{array}$$

Ex. 15. What decimal of a week is 3.74976 of a minute?

Dividing by the numbers which connect the denominations in order, by Rule 2, we have

$$\begin{array}{r|l} 60) & \underline{3.74976} \ \text{min.} \\ 24) & \underline{.062496} \ \text{hr.} \\ 7) & \underline{.002604} \ \text{day} \\ & .000372 \ \text{wk.} \end{array}$$

Ex. 16. Express .9904 of a guinea in the denomination of pounds.

$$\begin{array}{rl} & .9904 \ \text{gui.} \\ & 21 \\ \hline 20) & \underline{20.7984} \ s. \\ & 1.03992 \ £. \end{array}$$

Ex. 17. Reduce $15s.\ 6\frac{3}{4}d.$ to the decimal of a pound.

$$6\tfrac{3}{4}d. = 6.75d. = \frac{6.75}{12}s. = .5625s.$$

$$\therefore\ 15s.\ 6\tfrac{3}{4}d. = 15.5625s. = \frac{15.5625}{20}£. = .778125£.$$

The operation is usually exhibited thus,

$$\begin{array}{r|l} 4) & \underline{3.00q.} \\ 12) & \underline{6.75d.} \\ 20) & \underline{15.5625s.} \\ & .778125£. \end{array}$$

Ex. 18. Find the value of .9375 cwt.

$$\overset{\text{cwt.}}{.9375} = \overset{\text{qrs.}}{.9375} \times 4 = \overset{\text{qrs.}}{3.75},$$

$$\text{and } \overset{\text{qrs.}}{.75} = \overset{\text{lbs.}}{.75} \times 28 = \overset{\text{lbs.}}{21},$$

whence the required value is 3 qrs. 21 lbs.

This operation is usually written thus,

$$\begin{array}{rr} & .9375 \\ & \underline{4} \\ \text{qrs.} & 3.7500 \\ & \underline{28} \\ \text{lbs.} & 21.00 \end{array}$$

Ex. 19. Find the exact value of $.54\dot{3}£$.

$$\begin{array}{l} \ .5433 \text{ \&c.} \\ \underline{20} \\ s.\ 10.8666 \text{ \&c.} \\ \underline{12} \\ d.\ 10.3999 \text{ \&c.} = d.\ 10.4 \quad \text{(Art. 38.)} \\ \phantom{d.\ 10.3999 \text{ \&c.} = d.\ 10.}\underline{4} \\ \phantom{d.\ 10.3999 \text{ \&c.} =\ } q.\ 1.6 \end{array}$$

Or by means of the corresponding vulgar fractions, thus,

$$.54\dot{3}£ = £\frac{543-54}{900} \text{ (Art. 35, } c,) = £\frac{163}{300} = 10s.\ 10d.\ 1\frac{3}{5}q.$$

Ex. 20. How many bricks, each 9 inches square, will a floor contain which is 18 feet wide and 27 feet long?

The area of the floor $= 18 \times 27$ sq. ft. $= 216 \times 324$ sq. in., and the surface of each brick $= 9 \times 9$ sq. in.; we have, therefore, by dividing the former of these dimensions by the latter,

$$\frac{216 \times 324}{9 \times 9} = 24 \times 36 = 864, \text{ the required number.}$$

Ex. 21. How many bricks 9 inches long, $4\frac{1}{2}$ in. wide, and 3 in. thick, are contained in a wall which is 12 feet long, 6 ft. 9 in. high, and a brick and a half thick?

The solid content of the wall $= 144 \times 81 \times 13\frac{1}{2}$ cubic in.
.... of each brick $= 9 \times 4\frac{1}{2} \times 3$ cubic in.;

and, dividing the former by the latter, we have

$$\frac{144 \times 81 \times 13\frac{1}{2}}{9 \times 4\frac{1}{2} \times 3} = 144 \times 9 = 1296, \text{ the required number.}$$

Ex. 22. What is the length of a rectangular parallelopiped, the solid content of which is 120 cubic feet, the breadth 4 feet, and the depth 3 feet 9 inches?

Since the solid content of a parallelopiped is the continued product of its length, breadth, and depth; any one of these may be found by dividing the solid content by the product of the other two.

Hence the length of the given parallelopiped

$$= \frac{120}{4 \times 3\frac{3}{4}} = \frac{120 \times 4}{60} = 8 \text{ feet.}$$

Ex. 23. Find the content of a rectangular field, of which the length is 27 chains 50 links, and breadth 10 chains 5 links.

27 chains 50 links = 2750 links,
and 10 chains 5 links = 1005 links,

whence, the content of the field $= \overset{\text{links}}{2750} \times \overset{\text{links}}{1005} = \overset{\text{sq. links}}{2763750}$

$$= \frac{2763750}{100000} \text{ acres} = \overset{\text{acres}}{27.6375} = \overset{\text{ac.}}{27} - \overset{\text{roods}}{2} - \overset{\text{perches}}{22}$$

The operation may be performed thus,

$$\begin{array}{rr}
27 \text{ chains } 50 \text{ links} = & 2750 \text{ links} \\
10 \text{ chains } \ \ 7 \text{ links} = & \underline{1005} \text{ links} \\
 & 13750 \\
 & \underline{2750} \\
\text{acres} & 27.63750 \\
 & \underline{4} \\
\text{roods} & 2.5500 \\
 & \underline{40} \\
\text{perches} & \underline{22.00}
\end{array}$$

Ex. 24. How many seconds are there in the solar year, or in 365 days, 5 hours, 48 minutes, 51.6 seconds?

Ans. 31556931.6.

Ex. 25. How many miles, &c. are there in 364392 inches?

Ans. 5 mls. 6 fur. 2 yds.

Ex. 26. How many seven-shilling pieces are there in 1191£. 15*s.*? *Ans.* 3405.

Ex. 27. How many revolutions will a wheel, which is 7 ft. 6 in. in circumference, make in running 20 miles?

Ans. 14080.

Ex. 28. Reduce 4 days, 4 hours, 48 minutes, to the fraction of a week. *Ans.* $\frac{3}{5}$.

Ex. 29. Find the values of $\frac{3}{8}$ of a pound, and $\frac{4}{9}$ of a guinea.

Ans. 7*s.* 6*d.*, and 9*s.* 4*d.*

Ex. 30. Reduce 15 minutes 48.924 seconds to the decimal of a degree. *Ans.* $.2635\dot{9}$.

Ex. 31. Find the values of .0432 of a week, and $3.11680\dot{5}$ days. *Ans.* 7 h. 15 m. 27.36 s., and 3 d, 2 h. 48 m. 12 s.

Ex. 32. What is the content of a rectangular field, the length of which is 13 chains 75 links, and the breadth 12 chains 10 links? *Ans.* 16 acres 2 roods 22 perches.

Ex. 33. What is the height of a room containing $3822\frac{2}{9}$ cubic feet, its length being $21\frac{1}{2}$ feet, and breadth $17\frac{7}{9}$ feet? *Ans.* 10 feet.

Ex. 34. How many hours will there be in the years 1844 and 2200 respectively, by the calendar? *Ans.* 8784, and 8760.

ADDITION AND SUBTRACTION OF COMPOUND NUMBERS.

46. To add or subtract quantities consisting of different denominations, write the parts which are of the same denomination one under the other, and operate separately on each, beginning with those of the lowest denomination.

If the sum of a column be equal to, or exceed the number contained in an unit of the next superior order, put down the excess, if any, and add the number of the next higher denomination contained in that sum to the next column.

Also, if in any column the number to be subtracted exceed the subtrahend, add to the subtrahend as many as make one of the next higher denomination, instead of borrowing 10 as in abstract numbers, then put down the difference and carry one to the next number to be subtracted.

Ex. 1. Required the sum of the following quantities:

	Tons	cwt.	qrs.	lbs.	oz.	drs.
	17	5	3	24	13	10
	11	0	2	7	5	8
	20	19	0	27	14	3
	9	15	1	6	6	12
and	4	7	2	5	15	2
Ans.	63	8	2	16	7	3

Ex. 2. Required the difference of

	lbs.	oz.	dr.	sc.	gr.
	7	6	4	2	2
and	3	8	4	2	15
Ans.	3	9	7	2	7

Ex. 3. Required the sum and difference of $\frac{2}{3}$ of 5 guineas and $\frac{3}{4}$ of $\frac{7}{9}$ of a pound.

$\frac{2}{3}$ of 5 gui. = £3. 10*s.* and $\frac{3}{4}$ of $\frac{7}{9}$ £ = 11*s.* 8*d.*

Whence the required sum = £4. 1*s.* 8*d.*

and the difference = £2. 18*s.* 4*d.*

Ex. 4. Required the sum and difference of $.\dot{6}$ of a pound, and .875 of a shilling.

$.\dot{6}$£. = 13*s.* 4*d.*, and .875*s.* = 10½*d.*

Whence the sum =14*s.* 2½*d.*, and the difference =12*s.* 5½*d.*

Ex. 5. Find the difference between 17 lbs. 9 oz. 10 dwts. 21 gr., and 9 lbs. 10 oz. 19 dwts. 10 gr. of silver.

Ans. 7 lbs. 10 oz. 11 dwts. 11 gr.

Ex. 6. Collect into one sum 3 lbs. 5 oz. 7 dr. 2 scr. 16 gr. 13 lbs. 7 oz. 3 dr., and 9 lbs. 11 oz. 1 scr. 6 gr., apothecaries' weight. *Ans.* 27 lbs. 3 dr. 1 scr. 2 gr.

Ex. 7. Required the sum and difference of $\frac{1}{3}$ of a pound, and $\frac{2}{9}$ of a guinea.

Ans. The sum is 11*s.* 4*d.*, and the difference 2*s.*

Ex. 8. Required the sum and difference of .75 of a pound and $.\dot{5}$ of a guinea.

Ans. The sum is 1£. 6*s.* 8*d.*, and the difference 3*s.* 4*d.*

MULTIPLICATION AND DIVISION OF COMPOUND NUMBERS.

47. To multiply a compound by a simple or abstract number, place the multiplier under the number of the lowest denomination of the proposed quantity; find the product of this and the givén multiplier, and ascertain the number of units of the next higher denomination contained in it, put down the remainder, if any, and carry these units to the product of the multiplier and the number of the next higher order; then proceed with this as with the first product, and continue the process until the whole is multiplied.

To divide a compound by an abstract number, place the divisor and dividend as in division of abstract numbers, then find how often the divisor is contained in the number of the highest denomination in the quantity proposed, put down the quotient and reduce the remainder to the next inferior denomination, adding to it the number of the same denomination in the given quantity; repeat the division and proceed with the remainder as before; and continue the process until the whole is divided.

Ex. 1. Multiply 18£. 12*s.* $7\frac{1}{2}$*d.* by $\frac{5}{6}$.

Multiplying the given quantity by the numerator, and dividing the product by the denominator, we have

	£.	s.	d.
	18	12	$7\frac{1}{2}$
			5
6)	93	3	$1\frac{1}{2}$
	15	10	$6\frac{1}{4}$

Ex. 2. Divide 784£. 10s. into 48 parts.

$$6 \times 8 = 48 \begin{cases} 6)\ \underline{784 \quad 10} \\ 8)\ \underline{130 \quad 15} \end{cases}$$

	£.	s.	d.
6)	784	10	
8)	130	15	
	16	6	$10\frac{1}{2}$

Ex. 3. Multiply 3 tons 5 cwt. 2 qrs. by 38.

tons	cwt.	qrs.	
3	5	2	
		6	$6 \times 6 + 2 = 38$
19	13	0	
		6	
117	18	0	$= 36 \times$ the proposed quantity
6	11	0	$= 2 \times$ ....
124	9	0	$= 38 \times$ ....

Ex. 4. Divide 124 tons 9 cwt. by 38.

```
      tons  cwt.  t.  c.  q.
  38) 124    9 (3   5   2
      114
      ---
       10
        20
      ---
  38) 209 (5 cwts.
      190
      ---
       19
        4
      ---
  38)  76 (2 qrs.
       76
      ---
       ...
```

Ex. 5. Find the value of $\frac{2}{3}$ of 135£. 16s. $10\frac{1}{2}d$.

Ans. 90£. 11s. 3d.

Ex. 6. What is the eighteenth part of 21 acres 24 perches?
Ans. 1 acre 28 perches.

Ex. 7. Multiply 5 hrs. 1 min. 19.2 sec. by 23, and divide the product by 31. *Ans.* 3 hrs. 43 min. 33.6 sec.

THE COMPUTATIONS OF ARTIFICERS BY CROSS-MULTIPLICATION.

48. Cross Multiplication is a convenient method of computing small areas and solid contents, and is principally used by artificers to estimate the superficial or solid contents of work done.

The dimensions are commonly taken in feet, and parts decreasing in a twelve-fold ratio, termed primes, seconds, thirds, &c., which are distinguished by accents placed a little to the right above the numbers to which they belong; thus, 23 feet, 9 primes, 6 seconds, 4 thirds, &c , are written 23^f, $9'$, $6''$, $4'''$, &c.

It is also called Duodecimal Multiplication and Duodecimals; but the operations are not conducted in the duodecimal scale of notation, the digits of the several denominations not being connected with each other by the factor 12, though the denominations themselves are.

The rule for conducting the operations of Cross Multiplication may be enunciated as follows.

Rule. Write the terms of the multiplier under the corresponding terms of the multiplicand.

Multiply every term in the multiplicand, beginning at the lowest, by each term in the multiplier successively, beginning with the highest, divide each product which is not of the denomination of feet by 12, and place the remainder

under the multiplicand when the denomination of the multiplier is feet, one place removed to the right when it is primes, two places when it is seconds, three when it is thirds, &c., observing always to add the quotient to the next product.

The sum of the products thus obtained will be the one required.

The nature of the process will readily appear from the following considerations:

		ft. ft. sq. ft.
Feet × feet give square feet;	thus	$2 \times 5 = 10$
Feet × primes give primes;	...	$2 \times \frac{5}{12} = \frac{10}{12} = 10'$
Feet × seconds give seconds;	...	$2 \times \frac{5}{144} = \frac{10}{144} = 10''$
Primes × primes give seconds;	...	$\frac{2}{12} \times \frac{5}{12} = \frac{10}{144} = 10''$
Primes × seconds give thirds;	...	$\frac{2}{12} \times \frac{5}{144} = \frac{10}{1728} = 10'''$
&c..........	....	....

The terms of the given dimensions being linear feet and parts, descending to the right in a twelve-fold proportion, those of the resulting products are square feet and parts, descending to the right in the same proportion, the primes, seconds, &c. of the latter being $\frac{1}{12}$ sq. ft., $\frac{1}{144}$ sq. ft., &c.

If these products be multiplied by other linear dimensions, the result will be cubic feet and parts, descending in a twelve-fold proportion.

Ex. 1. Let it be required to find the area of a rectangular parallelogram whose adjacent sides are 8 ft. 6′ 9″ and 7 ft. 3′ 3″.

Multiplying 8 ft. 6′ 9″ by 7 ft. 3′ 3″ according to the rule, we have

	8f	6′	9″		
	7	3	3		
	59	11	3		
	2	1	8	3	
		2	1	8	3
sq. ft.	62	3′	0″	11‴	3⁗ = area.

Ex. 2. Let the solid content of a parallelopiped be required, the length of which is 2 ft. 3′, the width 1 ft. 7′, and the depth 9′ 4″.

	2f	3′		
	1	7		
	2	3		
	1	3	9	
sq. ft.	3	6′	9″	
		9	4	
	2	8	0	9
		1	2	3
Ans. cu. ft.	2	9′	3″	

Ex. 3. What is the area of a floor, its length being 15 ft. 9′ 8″ and width 12 ft. 5′? *Ans.* 196 sq. ft. 3′ 4″.

Ex. 4. How many square feet of glass are there in 4 windows, each measuring 5 ft. 7′ in height and 3 ft. 5′ in width? *Ans.* 76 sq. ft. 3′ 8″.

Ex. 5. The edges of a cube being 2 ft. 8 in., what is its solid content? *Ans.* 18 cu. ft. 11′ 6″ 8‴.

Ex. 6. What is the solid content of a piece of hewn timber, the length of which is 22 ft. 3 in., and the mean breadth

and thickness each 1 ft. 3 in.; the solid content being the continued products of the length, the mean breadth, and mean thickness? *Ans.* 34 cu. ft. 9′ 2″ 3‴.

Ex. 7. Find the solid content of a piece of rough timber, of which the length is 15 feet, and mean girth 82 inches; the solid content being the product of the length and the square of one-fourth of the mean girth?

Ans. 43 cu. ft. 9′ 3″ 9‴.

49. Brick-work being measured by the rod of $16\frac{1}{2}$ feet, and $272\frac{1}{4}$ square feet of brick-work, a brick and a half or three half-bricks thick, being reckoned a square rod, if the thickness of a wall be more or less than a brick and a half, it is necessary, in order to ascertain the amount of brick-work, to reduce its thickness to this standard by multiplying the superficial content of the side of the wall by the multiple or fraction which the given thickness is of a brick and a half.

Ex. 1. Required the amount of brick-work in a wall which is 363 feet long, 9 feet high, and $2\frac{1}{2}$ bricks thick.

$$\text{The given thickness} = \frac{5}{2} \text{ bricks} = \frac{5}{3} \text{ of 3 half-bricks,}$$

and therefore the amount of brick-work

$$= 363 \text{ ft.} \times 9 \text{ ft.} \times \frac{5}{3} = 5445 \text{ sq. ft.} = \frac{5445}{272\frac{1}{4}} \text{ sq. rds.} = 20 \text{ sq. rds.}$$

This evidently is increasing the side of the wall by the bricks taken from its thickness, whether by making it two-thirds higher, or two-thirds longer, or otherwise, the thickness of a brick and a half being preserved throughout.

Ex. 2. How many square rods, &c., are there in a wall which is 76 ft. 6 in. long, 7 ft. 4 in. high, and $3\frac{1}{2}$ bricks thick? *Ans.* 4 sq. rods, 220 sq. ft.

Ex. 3. A garden is enclosed by a wall, the mean length of which, found by measuring the outer and inner side, and taking half the sum of these dimensions, is 235 ft. 6 in., the height 9 ft. 4 in., and the thickness 3 bricks; how many square rods, &c., does it contain?

Ans. 16 sq. rods, 40 sq. ft.

RATIO AND PROPORTION.

50. By the term RATIO is meant the relation which exists between two quantities of the same kind, with respect to magnitude.

Hence the ratio of 6 to 3, written thus 6 : 3, denotes the magnitude of 6 with respect to 3, and is obviously equivalent to $\frac{6}{3} = 2$, the former term, called the *antecedent*, being double of the latter, which is called the *consequent*. Similarly, the ratios of 4 to 5, 3 to 7, 5 to 10, &c., may be expressed thus, 4 : 5, 3 : 7, 5 : 10, &c., or thus,

$$\frac{4}{5}, \frac{3}{7}, \frac{5}{10}, \text{ \&c., of which } 5 : 10 = \frac{5}{10} = \frac{1}{2} = 1 : 2.$$

A ratio is said to be a ratio of *greater* or *less inequality*, according as the antecedent is greater or less than the consequent, or as the fractions which represent them are greater or less than 1.

(*a*). It is manifest from Art. 10, that *the terms of any ratio may be multiplied or divided by the same number without altering its value.*

51. *A ratio of greater inequality is diminished and of less inequality increased by adding the same quantity to both its terms.*

First, let us take the ratio of greater inequality $7 : 3$, and add 2 to both its terms, so that it becomes $9 : 5$; then the former ratio $= \frac{7}{3} = \frac{35}{15}$, and the latter ratio $= \frac{9}{5} = \frac{27}{15}$; hence it is evident that the new ratio is less than the original one, (Art. 23, p. 17).

Secondly, if we take the ratio of less inequality $3 : 5$, and add 2 to both its terms so that it becomes $5 : 7$, we have the former ratio $= \frac{3}{5} = \frac{21}{35}$, and the latter ratio $= \frac{5}{7} = \frac{25}{35}$, in which case the new ratio is greater than the original one. See also Art. 29, p. 23.

52. PROPORTION consists in the equality of ratios. Thus the ratio of $3 : 4$ being equal to that of $6 : 8$, the four numbers 3, 4, 6, 8 constitute a proportion which is written thus, $3 : 4 = 6 : 8$, or thus, $3 : 4 :: 6 : 8$, and read thus, 3 is to 4 as 6 to 8; it may also be written fractionally, thus, $\frac{3}{4} = \frac{6}{8}$.

53. In the proportion $3 : 4 :: 6 : 8$, if we multiply the equal ratios $\frac{3}{4}$ and $\frac{6}{8}$ by 4×8 or 32, we have $3 \times 8 = 6 \times 4$, that is, *the product of the extreme terms = that of the mean terms:* and if we divide these equal products by 3, we have $8 = \frac{6 \times 4}{3}$, or *the fourth term = the product of the second and third divided by the first.*

(*a*). Hence, if the first three terms of a proportion be given, the fourth may be found; and it may be shewn, in the same manner, that either of the extremes may be obtained by dividing the product of the means by the other, and either of the means by dividing the product of the extremes by the other.

(*b*). Hence also it is obvious, that *the first and second*, or *the first and third terms of a proportion, may be divided by any common factor without altering the value of the fourth*, (Art. 10).

54. If we have two proportions, as

$$4 : 5 :: 12 : 15 \text{ and } 8 : 12 :: 2 : 3,$$

$$\text{or } \frac{4}{5} = \frac{12}{15} \text{ and } \frac{8}{12} = \frac{2}{3},$$

we have, by multiplying together the corresponding terms of these equals, $\frac{4 \times 8}{5 \times 12} = \frac{12 \times 2}{15 \times 3}$,

$$\text{or } 4 \times 8 : 5 \times 12 :: 12 \times 2 : 15 \times 3;$$

and *if the corresponding terms of any two proportions be multiplied together, the products thence arising will form another proportion.*

The same is true of any number of proportions.

This is called *compounding* the proportions.

55. In the proportion $3 : 4 :: 6 : 8$, or $\frac{3}{4} = \frac{6}{8}$, we have, by dividing 1 by each of these ratios, $\frac{1}{\frac{3}{4}} = \frac{1}{\frac{6}{8}}$, $\therefore \frac{4}{3} = \frac{8}{6}$,

or $4 : 3 :: 8 : 6$; hence we infer that *if the terms of any two equal ratios be inverted, the resulting ratios will be equal*, or in other words, *the four quantities will still constitute a proportion.*

56. In the proportion $3 : 7 :: 12 : 28$, by multiplying each of the equal ratios by that of $7 : 12$, thus

$$\frac{3}{7} \times \frac{7}{12} = \frac{12}{28} \times \frac{7}{12}, \text{ we have } \frac{3}{12} = \frac{7}{28},$$

$$\text{or } 3 : 12 :: 7 : 28;$$

Hence, *if the terms of a given proportion be taken alternately, they will constitute a proportion.*

57. Again, if we add 1 to the equal ratios 7 : 3 and 21 : 9, we shall have

$$\frac{7}{3}+1=\frac{21}{9}+1, \therefore \frac{7+3}{3}=\frac{21+9}{9},$$

$$\text{or } 7+3:3::21+9:9.$$

Hence *the ratio of the first term + the second to the second = that of the third term + the fourth to the fourth term.*

58. If we subtract 1 from the equal ratios 7 : 3 and 21 : 9, we shall have

$$\frac{7}{3}-1=\frac{21}{9}-1, \therefore \frac{7-3}{3}=\frac{21-9}{9},$$

$$\text{or } 7-3:3::21-9:9;$$

Hence *the excess of the first term above the second has to the second the same ratio that the excess of the third above the fourth has to the fourth.*

59. By inverting the terms of the equal ratios 7 : 3 and 21 : 9, Art. 55, and compounding the resulting proportion with 7 − 3 : 3 :: 21 − 9 : 9, Art. 54, we have

$$\frac{7-3}{3}\times\frac{3}{7}=\frac{21-9}{9}\times\frac{9}{21}, \therefore \frac{7-3}{7}=\frac{21-9}{21},$$

$$\text{or } 7-3:7::21-9:21,$$

$$\text{and } 7:7-3::21:21-9; \text{ (Art. 55.)}$$

Hence *the first term has to its excess above the second the same ratio that the third has to its excess above the fourth.*

60. From the proportion 7 : 3 :: 21 : 9, we have, in Art. 57, obtained the proportion 7 + 3 : 3 :: 21 + 9 : 9, and in Art. 58, 7 − 3 : 3 :: 21 − 9 : 9; now if we invert the terms of the proportion obtained in Art. 58, and com-

pound the resulting proportion with that obtained in Art. 57, which, it may be observed, is nothing more than dividing the former equal ratios by the corresponding ratios of the latter proportion, we have

$$\frac{7+3}{3} \times \frac{3}{7-3} = \frac{21+9}{9} \times \frac{9}{21-9}, \quad \therefore \frac{7+3}{7-3} = \frac{21+9}{21-9},$$

$$\text{or } 7+3 : 7-3 :: 21+9 : 21-9;$$

Hence *the sum of the first and second terms has to their difference the same ratio that the sum of the third and fourth has to their difference.*

61. From the same kind of reasoning it will follow, that *if there be any number of equal ratios whatever, any one of the antecedents is to its consequent as the sum of all the antecedents to the sum of all the consequents.*

62. The relative magnitudes of the quantities which constitute any proportion are evidently independent of the manner in which they are expressed; the terms of a proportion may, therefore, be abstract integers, vulgar or decimal fractions, or compound numbers.

63. The demonstrations which are given here are necessarily confined to particular instances, but the same reasoning may be applied in every case; and by adopting general algebraical symbols the demonstrations may be made general.

64. The terms Ratio and Proportion being used here in the same sense as they are in Geometry, are generally called *Geometrical Ratio* and *Geometrical Proportion.* The terms *Arithmetical Ratio* and *Arithmetical Proportion* also are sometimes employed to signify the relations of numbers in respect of their *Differences.*

SIMPLE PROPORTION, OR THE SINGLE RULE OF THREE.

65. The application of the doctrines of Ratio and Proportion to the solution of questions in which a fourth term is to be found from three that are given, constitutes what is commonly called the Single Rule of Three. There is, however, a variety of questions to which the doctrines of Ratio and Proportion are similarly applied, but which are generally classed under different heads, with appellations suitable to their nature.

In all questions of this kind, one of the given magnitudes, of which there are two of the same kind, has to the other the same ratio which the third given magnitude has to the one required; and from the proportion thus established the fourth or required term may be found by Art. 53.

Ex. 1. If 6 men reap 30 acres of corn in 10 days, how many acres will 8 men, working at the same rate, reap in the same time?

Here it is clear that the ratio of 6 men to 8 is the same as that of the number of acres which 6 men can reap in the given time, to the number which 8 men can reap in the same time: adopting, therefore, the notation of Art. 50, and proceeding to find the fourth term by Art. 52, we have

6 : 8 :: 30 : the number of acres required,
or 1 : 8 :: 5 : ....

$\therefore$ the required number $= 8 \times 5 = 40$.

Ex. 2. Find a number which shall have to 9 the same ratio which 20 has to 15.

Here we have 20 : 15 :: required number : 9,
or, Art. 55, 15 : 20 :: 9 : required number,
Art. 52, *b*, 1 : 4 :: 3 : ....

$\therefore$ the required number $= 4 \times 3 = 12$.

Or thus, $\frac{\text{required number}}{9} = \frac{20}{15}$, and multiplying these equal ratios by 9, we have

$$\text{the required number} = \frac{20 \times 9}{15} = 4 \times 3 = 12.$$

Ex. 3. If the pendulum of a clock vibrate 240 times in 4 minutes, how many times will it vibrate in an hour and three quarters?

The number of vibrations being the same in equal portions of time, we have the following proportion,

$$\overset{\text{m.}}{4} : \overset{\text{h.}}{1\tfrac{3}{4}} :: 240 : \text{the required number};$$

and expressing the terms of the former ratio in the same denomination previously to multiplying and dividing, we have

$$\overset{\text{m.}}{4} : \overset{\text{m.}}{105} :: 240 : \text{the required number},$$

$$\text{or, } 1 : 105 :: \ 60 : \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(Art. 52, } b.\text{)},$$

$$\therefore \text{ the required number} = 105 \times 60 = 6300.$$

Ex. 4. If $\frac{2}{3}$ of a ship be worth 3000£, what is $\frac{1}{6}$ of her value?

$$\frac{2}{3} : \frac{1}{6} :: 3000\text{£.} : \text{the value required};$$

whence the value of $\frac{1}{6}$ of the ship

$$= 3000\text{£.} \times \frac{1}{6} \div \frac{2}{3} = 500\text{£.} \times \frac{3}{2} = 750\text{£.}$$

Ex. 5. The hour and minute hands of a clock are in conjunction at 12 o'clock; at what instant between 3 and 4 o'clock are they together?

Since the hands move uniformly at rates in the ratio of 12 : 1, it is manifest that at the required time the spaces

moved over by them and indicated by the hours, must be in this ratio; we shall therefore have, between 3 and 4 o'clock,

$$36 + \text{required time} : \text{required time} :: 12 : 1,$$

$$\therefore\ 36 : \text{required time} :: 11 : 1, \text{ (Art. 58)},$$

$$\therefore \text{ the required time} = \tfrac{36}{11} \text{ hrs.} = \overset{\text{hrs.}}{3} \cdot \overset{\text{m.}}{16} \cdot \overset{\text{sec.}}{21\tfrac{9}{11}}. \text{ (Art. 53, } a.)$$

Ex. 6. A cistern is filled by a tap A in a quarter of an hour, and by a tap B in 25 minutes; also its whole content is discharged by a tap C in half-an-hour; in what time will it be filled if the three taps be all left open together?

Representing the content of the cistern by 1, we have

$\frac{1}{15}$ = the portion through A in 1 minute,
$\frac{1}{25}$ = ... B ....
$\frac{1}{30}$ = ... C ....

And it is evident that $\frac{1}{15}+\frac{1}{25}-\frac{1}{30}$ or $\frac{11}{150}$ is the portion of the cistern filled in 1 minute when the three taps are left open together; whence we have the following proportion,

$$\tfrac{11}{150} : \text{the whole content of the cistern} :: \overset{\text{m.}}{1} : \overset{\text{m.}}{13\tfrac{7}{11}}$$

the required time.

Obs. 1. In the preceding examples the proportion is termed *Direct.*

Ex. 7. If 8 men can complete a piece of work in 12 days, how many days will it take 6 men, working at the same rate, to complete the same?

Here, 8×12 = the number of days in which 1 man could do it, $\therefore \dfrac{8 \times 12}{6} = 16$, the number of days required for 6 men to do the same:

$$\text{or } \overset{\text{men}}{6} : \overset{\text{men}}{8} :: \overset{\text{days}}{12} : \overset{\text{days}}{16}.$$

Obs. 2. In this example, the time is increased as the number of men is diminished; or, in other words, the time varies inversely as the number of men, and the proportion is therefore termed *Inverse.*

Ex. 8. Find a number which shall have to 40 the ratio of 3.75 to 3. *Ans.* 50.

Ex. 9. The rateable value of a parish amounts to 1750£., and a rate of 32£. 16*s.* 6*d.* is levied; what is that in the pound? *Ans.* $4\frac{1}{2}$*d.*

Ex. 10. How much must a person spend in a week in order to lay by 30£. in a year, if his annual income be 251£.? *Ans.* 4£. 5*s.*

Ex. 11. If three-fifths of an estate be worth 75£., what is the value of the whole? *Ans.* 125£.

Ex. 12. If a stick, which stands perpendicularly 3 feet 6 inches, project a shadow 7 feet long, what is the perpendicular height of a small object whose shadow, at the same time, is 46 feet from the point immediately under it?

Ans. 23 feet.

Ex. 13. At what instant between 2 and 3 o'clock do the hour and minute hands of a clock point in directions exactly opposite? *Ans.* 2h. 43m. 38 $\frac{2}{11}$ sec.

Ex. 14. If 800 soldiers consume 5 barrels of flour in 6 days, how many will consume 15 barrels in 2 days?

Ans. 7200.

Ex. 15. If a besieged garrison have provisions for 21 days, at the rate of 20 ounces a day for each man, to what quantity must each man's allowance be reduced, in order that the provisions may last 30 days? *Ans.* 14 ounces.

PRACTICE.

66. Many questions involving the doctrines of Ratio and Proportion admit of an easy solution by means of aliquot parts, which for that reason generally come under the denomination of *Practice*.

Ex. 1. What is the value of 72 dozen at 1*s*. 7*d*. per dozen.

72*s*. = the price at 1*s*.

∴ $\frac{1}{2}$ of 72*s*. = 36*s*. = ... 6*d*.

and $\frac{1}{6}$ of 36*s*. = 6*s*. = ... 1*d*.

∴ 114*s*., or 5£. 14*s*. = the price at 1*s*. 7*d*.

The operation is usually written thus,

6*d*.	$\frac{1}{2}$	72
1	$\frac{1}{6}$	36
		6
	20)	114*s*.
		5£. 14*s*.

Ex. 2. Required the value of 1874 lbs. at 10$\frac{3}{4}$*d*. per lb.

6*d*.	$\frac{1}{2}$*s*.	1874	
3	$\frac{1}{2}$	937*s*.	= the price at 6*d*.
1$\frac{1}{2}$	$\frac{1}{2}$	468 - 6*d*.	= .... 3*d*.
$\frac{1}{4}$	$\frac{1}{6}$	234 - 3	= .... 1$\frac{1}{2}$*d*.
		39 - $\frac{1}{2}$	= .... $\frac{1}{4}$*d*.
	20)	1678 - 9$\frac{1}{2}$	
		83£. 18*s*. 9$\frac{1}{2}$*d*.	= the price at 10$\frac{3}{4}$*d*.

Ex. 3. Find the value of 5 cwt. 2 qrs. 14 lbs. at 2£. 5*s*. 6*d*. per cwt.

		£.	*s*.	*d*.	
2 qrs.	$\frac{1}{2}$	2 -	5 -	6	= price of 1 cwt.
				5	
		11 -	7 -	6	= .. .. 5 cwt.
14 lbs.	$\frac{1}{4}$	1 -	2 -	9	= ... . 2 qrs.
			5 -	8$\frac{1}{4}$	= .. ..14 lbs.
		12 -	15 -	11$\frac{1}{4}$	= .... 5 cwt. 2 qrs. 7 lbs.

Ex. 4. What is the value of 454 at 2*s*. 9*d*. each?

Ans. 62£. 8*s*. 6*d*.

Ex. 5. What is the value of $953\frac{1}{3}$ at $9\frac{3}{4}$*d*. each?

Ans. 38£. 14*s*. 7*d*.

Ex. 6. What is the value of 274 at 10*s*. 9*d*. each?

Ans. 147£. 5*s*. 6*d*.

Ex. 7. What is the value of 1834 at 27*s*. 6*d*. each?

Ans. 2521£. 15*s*.

Ex. 8. What will be the amount of rate on an assessment of 37259£. 10*s*., at $10\frac{1}{2}$*d*. in the pound?

Ans. 1630£. 2*s*. $0\frac{3}{4}$*d*.

Ex. 9. What is the dividend on 30£. 15*s*., at 3*d*. in the pound? *Ans*. 7*s*. $8\frac{1}{4}$*d*.

Ex. 10. What is the value of 11 cwt. 3 qrs. 8 lbs., at 5£. 7*s*. 6*d*. per cwt.? *Ans*. 63£. 10*s*. $9\frac{1}{2}$*d*.

Ex. 11. What is the yearly rent of 1200 acres, 3 roods, at 28*s*. 6*d*. per acre? *Ans*. 1711£. 1*s*. $4\frac{1}{2}$*d*.

Ex. 12. What will be the expense of carpeting a room which is $32\frac{1}{2}$ feet long and $23\frac{3}{4}$ wide, at 5*s*. 6*d*. a square yard? *Ans*. 23£. 4*s*. $11\frac{3}{4}$*d*.

COMPOUND PROPORTION.

67. Under this head are comprehended questions which require one or more repetitions of the process made use of in simple proportion, or, which is equivalent, a compounding of the proportions. When the question involves two simple proportions, it is also said to belong to the Double Rule of Three.

Ex. 1. If a family of 9 persons spend 300£. in 8 months; how much money will serve 17 persons 11 months, at the same rate of expenditure?

This question suggests the following proportions,

$\overset{\text{pers.}}{9} : \overset{\text{pers.}}{17} = \overset{\text{£.}}{300}$: the sum required by 17 persons in 8 months,

and $\overset{\text{mo.}}{8} : \overset{\text{mo.}}{11}$ = the sum required by 17 in 8 months : the sum they require in 11 months.

Such terms as the fourth of the first proportion, and the third and fourth of the second proportion, are usually represented by symbols, thus

$$9 : 17 = 300 : x,$$
$$\text{and } 8 : 11 = x : y;$$

we have, therefore, by compounding these proportions, according to Art. 54,

$$8 \times 9 : 11 \times 17 = 300 \times x : x \times y,$$
$$\text{or } 72 : 187 = 300 : y, \text{ (Art. 50)},$$

whence y, the required sum, $= \dfrac{187 \times 25}{6} = 779$£. $3s.$ $4d.$

Here, by considering 8×9 and 11×17 as causes, and 300£. and the sum required as effects, we have their ratios equal.

It is obvious that the question might also have been solved by finding from the first proportion the sum required by 17 persons in 8 months, and then substituting that sum in the second proportion, in which the fourth term is the sum required.

Ex. 2. If 30 bushels of corn serve 3 horses for 50 days, how many days will 150 bushels serve 30 horses, at the same rate of allowance? *Ans.* 25.

Ex. 3. A wall 700 yards long was to be built in 29 days; at the end of 11 days 18 men had built 220 yards of it, how many additional men was it then necessary to engage to work at the same rate, in order that the wall might be completed in the given time? *Ans.* 6.

Ex. 4. If 14 men, working 9 hours a day, dig a trench 140 yards long, 2 yards wide, and 2 deep, in 4 days; in how many days will 36 men, working at the same rate for 8 hours a day, dig a trench 280 yds. long, 4 yds. wide, and 2 deep?

Ans. 7.

INTEREST.

68. Interest is the charge made for the loan or use of any sum of money for any length of time, and is generally estimated at so much *per cent per annum*. The money lent is called the *Principal*, the interest of 100£. for 1 year the *Rate per cent.*, and the aggregate of the sum lent, together with its interest for any time, the *Amount*.

Interest is said to be *simple* when it is paid for the principal only, and *compound* when paid for the amount as it becomes due.

69. To ascertain the *simple* interest of any given sum for a given time, we obviously have the following proportion,

100£. : the given sum = the rate per cent. : the interest for one year,

whence the interest for one year

$$= \frac{\text{the given sum} \times \text{the rate per cent.}}{100},$$

and this multiplied by the time will give the interest required.

Ex. 1. What is the interest and amount of 537£. 10*s*. for 3 years and 9 months at 4 per cent. per annum?

£.	*s*.	
537	10	
	4	
£21.50	0	= interest for 1 yr., pointing off two digits to the right, being equivalent to dividing by 100.
20		
s.10.00		

		£.	s.	d.	
6 mo.	½	21	10		= interest for 1 yr.
			3		
		64	10		= .... 3 yrs.
3	½	10	15		= .... 6 mo.
		5	7	6	= .... 3 mo.
		80	12	6	= .... 3 yrs. 9 mo.

£.	s.	d.	
80	12	6	= interest required
537	10	0	= principal
618	2	6	= amount.

Or, expressing the given sum, the *Ratio*, which is $\frac{\text{the rate per cent.}}{100}$, and the time decimally, the interest may be found thus,

$$\begin{array}{rl} £. & \\ 537.5 & \\ .04 & \text{ratio} \\ \hline 21.500 & \\ 3.75 & \text{time} \\ \hline 1075 & \\ 1505 & \\ 645 & \\ \hline £80.625 & \\ 20 & \\ \hline s.\ 12.500 & \\ 12 & \\ \hline d.\ 6.0 & \end{array}$$

Ex. 2. Find the simple interest and amount of 500 guineas for 1 year at $3\frac{1}{2}$ per cent. per annum.

Ans. 18£. 7*s.* 6*d.*, and 543£. 7*s.* 6*d.*

Ex. 3. What is the interest of 63£. 15*s.* for $1\frac{1}{4}$ year, at $4\frac{1}{2}$ per cent. per annum? *Ans.* 3£. 11*s.* $8\frac{1}{2}$*d.*

Ex. 4. What is the amount of 500£. in $4\frac{1}{4}$ years, at $4\frac{3}{4}$ per cent. per annum? *Ans.* 600£. 18*s.* 9*d.*

The interest for any number of days is generally calculated from the following proportion,

days
365 : given number of days = int. for 1 yr. : int. for those days, and when the time consists of both months and days, a more correct result may be obtained by finding the interest of the months and days, reduced to days, by the above proportion.

70. To find the amount of a sum of money in a given time at *compound* interest, first find the amount for one year, as in simple interest; then considering this as a new principal, find its amount for the second year, and so on for any given length of time.

Ex. 1. What is the amount of 300£. for $2\frac{1}{2}$ years, at $3\frac{1}{2}$ per cent. per annum?

$$
\begin{array}{rl}
\text{£.} & \\
300 & \\
.035 & \text{ratio} \\
\hline
10.500 & = \text{interest for 1st year.} \\
300 & \\
\hline
310.5 & = \text{1st year's amount.} \\
.035 & \\
\hline
10.8675 & = \text{interest for 2nd year.} \\
310.5 & \\
\hline
321.3675 & = \text{2nd year's amount.} \\
.035 & \\
\hline
2)\ 11.2478625 & = \text{interest for 3rd year.} \\
\hline
5.62393125 & = \text{interest for } \tfrac{1}{2} \text{ the 3rd year.} \\
321.3675 & \\
\hline
\text{£}326.99143125 & = 326\text{£. } 19s.\ 9d.\ 3.774q. = \text{am}^{t}.\ \text{req}^{d}.
\end{array}
$$

Ex. 2. What is the amount of 317£. 10*s.* for 2 years, at 3 per cent. per annum, compound interest?

Ans. 336£. 16*s.* 8*d.* 2.32*q.*

Ex. 3. What is the amount of 413£. 17*s*. 9*d*. for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum, compound interest?

Ans. 462£. 2*s*. 10*d*. 3.148...*q*.

Ex. 4. What is the amount of 550£. 10*s*. in a year and a half, at 4 per cent. per annum, compound interest, supposing the interest due half-yearly? *Ans*. 584£. 3*s*. 10*d*. 3.203...*q*.

Ex. 5. What is the amount of 315£. 10*s*. in a year and three quarters, at 5 per cent. per annum, compound interest, supposing the interest due half-yearly?

Ans. 344£. 1*d*. 1.702...*q*.

COMMISSION, BROKERAGE, AND INSURANCE.

71. Commission, Brokerage, and Insurance, being charges made at certain rates per cent., are calculated in the same manner as the interest for one year at the same rate.

Ex. 1. What is the commission on 500 guineas at $\frac{3}{8}$ per cent.? *Ans*. 1£. 19*s*. $4\frac{1}{2}$*d*.

Ex. 2. What is the brokerage of 1549£. 10*s*. at $2\frac{1}{2}$ per cent.? *Ans*. 38£. 14*s*. 9*d*.

Ex. 3. What premium must be paid for insuring 3120£. at $1\frac{1}{2}$ per cent.? *Ans*. 46£. 16*s*.

Ex. 4. What is the insurance, at $10\frac{3}{4}$ per cent., of a ship and cargo valued at 4500£.? *Ans*. 483£. 15*s*.

DISCOUNT.

72. Discount is an allowance made for the payment of money before it becomes due.

If the discount be subtracted from any proposed sum, the remainder is termed its *Present worth*, being that sum which, with interest at the same rate, would amount to the given

I 3

sum at the time it becomes due. Thus 100£. is the present worth of 105£. due at some future time, discount being made at 5 per cent.

To find the discount of any sum we obviously have the following proportion; the amount of 100£. for the given rate and time : the given sum :: the interest of 100£. for the given time : the discount of the given sum. And *to find the present worth*, we have the following proportion; the amount of £100. for the given rate and time : the given sum :: £100. : the present worth.

Instead however of finding the present worth by this proportion, the usual practice is to deduct from the sum due at a future time the interest upon it. Thus if the present value of 100£. due a year hence, be required of a banker, at 5 per cent. discount, he will pay but 95£; now the interest of 95£. for one year is only 95*s.*, there is, therefore, a gain to the Banker of nearly $\frac{1}{4}$ per cent, the sum which he would have to pay, by the proportion given above being 95£. 4*s.* 9*d.* $\frac{4}{7}$q.

Obs. Since the discount taken from the proposed sum gives the present worth, and the present worth taken from the proposed sum gives the discount, either of these being found the other may be obtained by subtraction.

Ex. 1. What are the present worth and discount of 651£. 13*s.* 4*d.*, due 5 months hence, at $4\frac{1}{2}$ per cent. per annum?

First, to find the amount of £100 for 5 months, we have

		£.	s.	d.			
4mo.	$\frac{1}{3}$	4	10	0	=	interest of £100. for 12mo.	
1	$\frac{1}{4}$	1	10	0	=	....	4 ...
		0	7	6	=	....	1 ...
		1	17	6	=	....	5 ...
		100	0	0			
		101	17	6	=	amount of 100£ for 5 months.	

Whence,

$101£.\ 17s.\ 6d : 100£. :: 651£.\ 13s.\ 4d. : 639£.\ 13s.\ 5\frac{1}{4}d.\ \frac{145}{163}$.

	£.	s.	d.	
Also,	651	13	4	= proposed sum
	639	13	$5\frac{1}{4}\ \frac{145}{163}$	= present worth
	11	19	$10\frac{1}{2}\ \frac{18}{163}$	= discount.

Ex. 2. What sum ought to be discounted, at the rate of 5 per cent. per annum, for the present payment of 700£. due 18 months hence? *Ans.* $48£.\ 16s.\ 8\frac{3}{4}d.\ \frac{31}{43}$.

Ex. 3. Determine accurately the present worth and discount of 140£. due 9 months hence, at 4 per cent per annum? *Ans.* $135£.\ 18s.\ 5\frac{1}{4}d.\ \frac{45}{103}$, and $4£.\ 1s.\ 6\frac{1}{2}d.\ \frac{58}{103}$.

Ex. 4. How much would a banker deduct on the present payment of 500£. due 15 months hence, with discount at 5 per cent. per annum? *Ans.* 31£. 5s.

Ex. 5. By how much does the interest of 1073£ exceed the discount for 12 months, at 5 per cent?

Ans. $2£.\ 11s.\ 1\frac{1}{7}d.$

EQUATION OF PAYMENTS.

73. The Equation of Payments is the finding of the proper time for the payment of two or more sums of money due at the ends of different periods.

Here it is evident that in order that neither party may be the loser, the interest of the sums due before the equated time must be equal to the discount of those due after that time; but in practice it is usual to make the interest of the sums due before the equated time equal to the interest of those due after that time.

Hence, since, on the latter supposition, the terms of the ratio each debt × its time : the sum of the debts × the equated

time are manifestly equal, the common method is to multiply each debt by its time, and to divide the sum of the products thus obtained by the sum of the debts; regarding the quotient as the equated time.

Ex. Suppose that of a debt of 1000£, 400£ becomes due 3 months hence, and the remainder 8 months hence, find the equated time for paying the whole.

By the common method,

$$\text{the equated time} = \frac{400 \times 360 + 0 \times 8}{1000} = \frac{60}{10} = 6 \text{ months.}$$

A method of ascertaining the true equated time will be found in the Algebra.

STOCKS.

74. The different national stocks, or public funds, are named according to the terms on which they were raised: thus the stock which pays 3 per cent. per annum to the holders is called 3 per cents; that which pays 4 per cent. is called 4 per cents., and so on. Transfers of stock are made at so much per cent., the price fluctuating according to the demand: thus when the price of any particular stock is said to be $83\frac{7}{8}$, 83£. 17*s.* 6*d.* must be paid for 100£. of this stock; or if the price be $84\frac{3}{8}$, 84£. 7*s.* 6*d.* must be paid for each 100£. Stock is generally transferred by means of a broker, and the brokerage is added to the price of a purchase, but deducted from a sale of stock.

Ex. 1. What must be given for 1500£. stock at $85\frac{1}{2}$, brokerage being $\frac{1}{8}$ per cent?

First, $85\frac{1}{2} + \frac{1}{8} = 85\frac{5}{8} = 85.625$; then,

from the nature of the question, we have the following proportion,

Stock	Stock	£.	£.	s.	d.	
100£. :	1500£. ::	85.625 :	1284	7	6,	the sum required.

Or, by Practice;

		£.				
50	$\frac{1}{2}$	1500			= price at £100	
25	$\frac{1}{2}$	750			=	50
10	$\frac{1}{5}$	375			=	25
$\frac{4}{8}$	$\frac{1}{20}$	150			=	10
$\frac{1}{8}$	$\frac{1}{4}$	7	10		=	$\frac{4}{8}$
		1	17	6	=	$\frac{1}{8}$
		1284	7	6	=	$85\frac{5}{8}$

Ex. 2. What sum will purchase 4000£. stock at $63\frac{5}{8}$, brokerage being $\frac{1}{8}$ per cent.? *Ans.* 2550£.

Ex. 3. How much stock at $90\frac{3}{8}$, will 2353£. purchase, brokerage being $\frac{1}{8}$ per cènt.? *Ans.* 2600£.

Ex. 4. What sum, including $\frac{1}{8}$ per cent. for brokerage, must be invested in the 3 per cents. at $90\frac{7}{8}$, to produce an income of 500£. a year? *Ans.* 15166£. 13*s.* 4*d.*

Ex. 5. What rate of interest does money invested in the 3 per cents. at 90 pay, brokerage being included in the price? *Ans.* $3\frac{1}{3}$ per cent.

BANKRUPTCY.

75. (*a.*) *The shares of the creditors, when the value of the effects is known,* may be computed by the following proportion;—the sum of the debts : the value of the effects : : the debt of each creditor : his share.

(*b.*) *How much in the pound a bankrupt's estate will pay,* may be ascertained by this proportion;—the value of his effects : the amount of his debts : : 1£. : the sum required.

(c.) *To compute the sum a creditor will receive at a certain composition or dividend,* we have the proportion

1£ : its dividend : : each creditor's debt : his share, which may easily be solved by "*Practice.*"

PARTNERSHIP.

76. (a.) *The respective shares of two or more persons having a joint stock, or common interest in a mercantile or other concern,* may be determined by the following proportion ;— the whole stock or capital : the whole gain or loss : : each proprietor's share : his share of the gain or loss.

(b.) *When the stocks of the individual proprietors are employed for different times, or under different circumstances,* their shares of the profit or loss may be determined from this proportion ;—the sum of the products arising from multiplying together each proprietor's stock and time : the whole gain or loss : : each individual product : the corresponding share of the gain or loss.

THE DOCTRINE OF EXCHANGE.

77. By Exchange is to be understood the act of bartering the money of one place or country for that of another. Its operations therefore consist in ascertaining by means of a simple proportion what sum of money of one country is equivalent to any given sum of another, according to some settled rate of commutation.

The course of Exchange is the variable sum of the money of one place which is given for a fixed sum of the money of another: and the *Par of Exchange* is the sum of money which is equal to that fixed sum.

Agio is the difference per cent. in some foreign places, between the *Bank* and the *Current* money; the former being finer or purer than the latter.

A Bill of Exchange is a written instrument by means of which exchanges are effected.

Usance is the usual time allowed for the payment of bills of Exchange; it varies in different countries.

At *Amsterdam* exchange is made by the pound sterling, for which Amsterdam gives to London a variable number of florins, from 11 to 13, more or less.

16 pennings = 1 stiver, and 20 stivers = 1 florin.

In *Paris* Exchange is made by the pound sterling, for which Paris gives 24 or 25 francs.

10 centimes = 1 decime, 10 decimes = 1 franc = 1 $\frac{1}{80}$ livre.

At *Lisbon* Exchange is by the milree, for which London gives from 45 to 55 pence, more or less.

1000 rees = 1 milree.

To change Bank into Current money,

100 : bank = 100 + the agio : currency.

To change Current money into Bank,

100 + the agio : currency = 100 : bank.

Ex. 1. How many florins, &c. must be received for a bill of 572£. 14*s.* sterling, remitted from London to Amsterdam, exchange being at 11 florins 10 stivers?

Sterling	Sterling		Flo.	Sti.		Flo.	Sti.
1£. :	572. 7£	=	11 -	10 :		6586 -	1.

Ex. 2. How much sterling must be paid in London to receive in Paris 5740 francs 14 centimes, exchange at 24 francs 22 centimes? *Ans.* 237£.

Ex. 3. How many milrees may be drawn at Lisbon for a bill of 284£. 8*s*. 6*d*. paid in London, exchange being at 3*s*. 10½*d*. per milree? *Ans*. 1468.

Ex. 4. What is the course of exchange between London and Lisbon, when 1512 milrees are received for 315£.? *Ans*. 50*d*.

Ex. 5. Change 364 florins 10 stivers bank into current money, agio 3½ per cent. *Ans*. 377 flo. 5 sti. 2. 4 pen.

78. The *Arbitration, or Comparison* of Exchanges, is the determining of such a rate of Exchange, called the ***Par of Arbitration***, between any number of places, as shall correspond with any assigned rates between each of them and another place. After computing this par of arbitration, by comparing it with the present course of Exchange, a person knows how to remit his money with the greatest advantage.

Arbitration is styled *simple* or *compound*, according as *three* or *more* places are concerned.

Further information on this subject, may be found in Dr. Kelly's *Universal Cambist*.

2

THE PRINCIPLES

OF

ALGEBRA,

CONTAINING THE MOST USEFUL PORTION OF THE SUBJECT, INCLUDING THAT WHICH IS REQUIRED FOR THE ORDINARY DEGREE OF B.A. AT THE UNIVERSITY OF CAMBRIDGE.

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CONTENTS.

ALGEBRA.

DEFINITIONS AND NOTATION.

Art. 1. ALGEBRA is the science of general mathematical reasoning by means of *Signs* and *Symbols;* the symbols being employed to represent magnitudes or quantities, and the signs to express their mutual dependence upon each other.

2. The symbols employed are the letters of the English or Roman, and Greek alphabets, and the relations of the quantities which they represent, or their mutual dependence, are expressed by connecting the symbols principally by the signs which are used in Arithmetic.

In this manner algebraical expressions are formed, which through various transformations are converted into general available formulæ; and thus, whilst Arithmetic *effects* the solution of a particular problem, Algebra *indicates*, in general terms, the solutions of all questions under the same or similar circumstances.

3. In the solution of problems, the letters from the beginning of the alphabet are usually selected to represent the known quantities or data, and the letters from the end of the alphabet the unknown quantities, or quæsita.

4. Quantities having the sign *plus*, +, before them, are termed *positive*, and sometimes *additive* or *affirmative;* and those which have *minus*, –, prefixed to them, *negative* or *subtractive.*

5. In Arithmetic the signs + and – are used to indicate the inverse operations of Addition and Subtraction only; but in Algebra they are employed also to denote contrariety of direction, heat and cold, to distinguish forces pushing from forces pulling, or any other opposite qualities or affections.

The extended use thus made of these signs frequently produces negative results, which have no meaning in Arithmetic, (since there is no quantity less than zero or 0,) but which in Algebra are interpreted according to the suppositions previously made.

To illustrate this, let a and b represent two forces acting in opposite directions; then, if a be positive b will be negative, and vice versâ; and their result will be represented by their algebraical sum (see Art. 41,) which is $a-b$ or $b-a$, according as a is positive or negative: suppose a to be positive, and b to be equal to twice a, then their result will be $a-b$ or $a-2a$, which is equivalent to $-a$, a force equal to a, but acting in the opposite direction. Now b being greater than a, the expression $a-b$ has no meaning in Arithmetic, whilst in Algebra its interpretation is obvious from the assumptions previously made.

Hence zero, which in Arithmetic is the absolute minimum, is considered in Algebra the common limit of positive and negative quantities; which, by such assumptions, are made to follow the order of the series

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

6. When no sign is prefixed to a leading symbol or number, $+$ is understood; for in writing an algebraical expression, if the leading symbol be positive the sign is usually omitted: thus $+a$, $+a+b$, $+a-b+c$, are written a, $a+b$, $a-b+c$.

7. The sign $\pm$ is used to signify *plus or minus*.

8. The sign $\sim$ is sometimes used to denote the *difference* of the quantities between which it is placed, when it is not known which of them is the greater: thus $a \sim x$ denotes the excess of a above x or of x above a, according as a is greater or less than x; and $a \ \underset{\sim}{+} \ x$ signifies the *sum or difference* of a and x.

9. Multiplication is indicated by the sign $\times$, read *into*, as in Arithmetic, and also by a full point, and the quantities between which they are placed are called *Factors*. The signs of Multiplication, however, are generally omitted in Algebra: thus, $a \times b \times c$ or $a.b.c$ is usually written abc.

But it may be observed that the sign $\times$ should always be inserted between two *numerical* factors: thus the product 4×7 will not be represented by 47; and if the full point be used, it may be confounded with the Decimal Point.

10. The multiples of quantities, such as twice x, three times x, &c. are expressed by placing the corresponding numbers, or their representatives, before them, thus, $2x$, $3x$,...nx; and the numbers or symbols thus prefixed are called their *Coefficients*.

In the product $3abx$., in which 3 is the numerical coefficient of abx, $3a$ may be considered the coefficient of bx, and $3ab$ the coefficient of x.

When no coefficient is prefixed, 1 is understood.

11. Division, as in Arithmetic, is indicated by the sign $\div$, or by writing the dividend over the divisor, and in this latter case the expression is called a fraction: thus the division of a by b is represented by $a \div b$, or by the fraction $\frac{a}{b}$.

12. A factor of any product is called also a *Measure* of that product, since it divides it without a remainder: and when two or more quantities have a factor which is common to them, the common factor is called also a *Common Measure* of those quantities, since it divides each of them without a remainder.

13. Quantities which are incommensurable, or which have not a common factor, are said to be *prime to each other.*

14. A quantity is called a *Common Multiple* of two or more others, when it exactly contains each of them any number of times, or when each of them divides it without a remainder.

15. An algebraical expression consisting of only one term, is called a *Simple*, or *Monomial Quantity*, as a, ab, $-abx$, $\frac{a}{b}$, &c.

A *Binomial* consists of two terms connected by the signs $+$ or $-$; as $a+b$, $a-b$, $ax+by$, &c.

A *Trinomial* consists of three terms connected by the signs $+$ or $-$; as $a+b+c$, $ax-by+cz$, &c.

Expressions consisting of many, or of an undefined or undetermined number of terms, connected in the same manner, are called *Multinomials* or *Polynomials*.

16. All expressions consisting of more than one term connected by the signs $+$ and $-$ are called *Compound Quantities.*

17. Compound quantities are further connected by means of the *Vinculum* $\overline{\quad}$, or by *brackets* ( ) or $\{\ \}$; such a connexion indicating that the quantities beneath the vinculum or within the brackets are to be taken collectively; thus

$$a+b-(c-d) \text{ or } a+b-\overline{c-d}$$

indicates the subtraction of $c-d$ from $a+b$;

$$\{a+b-(c-d)\}\,(x+y) \quad \text{or} \quad \overline{a+b-\overline{c-d}}\,.\,\overline{x+y}$$

indicates the product of $a+b-(c-d)$ and $x+y$.

18. If in Multiplication the same quantity be repeated any number of times, the product is usually expressed by placing a little to the right above the quantity, the number which represents how often it is repeated; thus, $a \times a$, or aa, is written a^2, and is called the *second power* or *square* of a; aaa is written a^3, and is called the *third power* or *cube* of a; $aaaa$, where the symbol is four times repeated, is written a^4, and is called the *fourth power* of a; and if the symbol be repeated any number (n) times in the expression $aaa \ldots$, it is written a^n, and is called the n^{th} power of a; and the number, or its representative, indicating the number of times the quantity is repeated, is called the *Index* or *Exponent* of the power.

When a symbol has no index or exponent, 1 is understood.

19. The raising of a quantity to any proposed power is called *Involution.*

20. One quantity is said to be the square, cube, or n^{th} root of another, when its square, cube, or n^{th} power gives the other quantity.

The square, cube, or n^{th} root, of a is denoted thus: $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[n]{a}$, (the numeral 2 being omitted in expressing the square root); and the sign $\sqrt{}$ is called the *Radical Sign.*

From the definition of a root it follows, that

$$\sqrt{a^2}=a, \quad \sqrt[3]{a^3}=a, \quad \sqrt[4]{a^4}=a, \ldots \sqrt[n]{a^n}=a.$$

These roots are expressed also by fractional indices; thus

$$a^{\frac{1}{2}}, \quad a^{\frac{1}{3}}, \quad a^{\frac{1}{4}}, \quad a^{\frac{1}{n}},$$

represent the square, cube, fourth, and n^{th} root of a respectively;

$$a^{\frac{3}{2}}, \quad a^{\frac{4}{3}}, \quad a^{\frac{n}{m}},$$

represent the square root of the cube of a, the cube root of the fourth power of a, and the m^{th} root of the n^{th} power of a respectively.

The consistency of the latter notation with the former, and with the definition of a root, will appear from the consideration, that

$$a^{\frac{2}{2}}=a, \quad a^{\frac{3}{3}}=a, \quad a^{\frac{4}{4}}=a, \ldots a^{\frac{n}{n}}=a.$$

The m^{th} root of the n^{th} root of a is denoted by $\sqrt[m]{\sqrt[n]{a}}$ or $a^{\frac{1}{mn}}$.

21. The extracting of roots is called *Evolution.*

22. If the roots indicated cannot be exactly determined, the quantities are called *Irrational,* or *Surds;* and if the radical sign be prefixed to a negative quantity, as $\sqrt{-a}$, it is also called an *Imaginary* or *Impossible* quantity.

23. An algebraical expression is said to be of the first, second, third, or n^{th} *Degree* or *Order,* and sometimes of one, two, three or n *Dimensions,* according as there are one, two, three

or n symbols involved in its terms; thus a is of the first degree, aa or a^2 and ab are of the second degree, the terms of the expression

$$a^3 + a^2x + ax^2 + x^3,$$

are of the third degree; and similarly

$$x^n,\ x^{n-1}y,\ x^{n-2}y^2,\ \&c.$$

are of the n^{th} degree, the degree of a product being found by taking the sum of the indices of the symbols involved in it.

A numerical coefficient is not considered to affect the degree of a quantity.

24. The terms of an algebraical expression are said to be *homogeneous* when the degree or order is the same in each: thus the terms of the expression

$$a^3 - 3a^2x + 3ax^2 - n^3$$

are homogeneous.

25. Compound quantities are said to be arranged according to the powers or dimensions of any symbol involved in them, when the indices of that symbol occur in the order of their magnitudes, either increasing or decreasing.

Thus, the terms of

$$a^3 - 3a^2x + 3ax^2 - x^3$$

are arranged according to descending powers of a and ascending powers of x.

26. *Similar* or *like* algebraical quantities are such as differ only in their signs and numerical coefficients: thus a and $-3a$, $-2x^2$ and $3x^2$, $7abc$ and $-5abc$, x^n and $-2x^n$, are pairs of similar or like quantities.

27. *Unlike* quantities differ in the symbols, or their exponents; thus $a, -b, a^2b, ax^2, -ax^{n-1}, 7ax^n$, are unlike quantities.

28. The sign $=$, as in Arithmetic, signifies that the quantities between which it is placed are *equal;* and the equality thus expressed is called an *Equation.*

29. The signs $>$ and $<$ are used to express the *inequality* of the quantities between which they are placed; $>$ is read *greater than*, and $<$ *less than*, the angular point being always contiguous to the smaller quantity.

30. *Therefore*, or *consequently* is expressed by $\therefore$.

Because, or *since*, by $\because$; the two dots being placed uppermost.

31. Points are made use of, as in Arithmetic, to denote Ratio and Proportion.

32. The following Examples will serve to illustrate the method of Algebraical Notation.

Let $a=5$, $b=4$, $c=3$, $d=2$; and $e=1$;
then $a-b+c-d=5-4+3-2=2$.

$3a-2b+cd=3\times5-2\times4+3\times2=13$.

$a+b-(c-d)=5+4-(3-2)=9-1=8$.

$(a-c)(b+d)=(5-3)(4+2)=2\times8=16$.

$$\frac{a+b+c}{16-(d+e)}=\frac{5+4+3}{16-(2+1)}=\frac{12}{13}.$$

$a^2=5\times5=25$,

$(d+e)^3=(2+1)^3=3\times3\times3=27$.

$\sqrt{a+b}=\sqrt{5+4}=\sqrt{9}=\sqrt{3\times3}=\sqrt{3^2}=3$:

or $(a+b)^{\frac{1}{2}}=(5+4)^{\frac{1}{2}}=9^{\frac{1}{2}}=(3\times3)^{\frac{1}{2}}=3^{\frac{2}{2}}=3$.

$\sqrt[3]{(a+c)}=\sqrt[3]{(5+3)}=\sqrt[3]{8}=\sqrt[3]{(2\times2\times2)}=\sqrt[3]{2^3}=2$;

or $(a+c)^{\frac{1}{3}}=(5+3)^{\frac{1}{3}}=8^{\frac{1}{3}}=(2\times2\times2)^{\frac{1}{3}}=2^{\frac{3}{3}}=2$.

AXIOMS.

33. Things which are equal to the same are equal to each other.

34. If the same or equal quantities be added to equal quantities, the sums will be equal.

35. If the same or equal quantities be taken from equal quantities, the remainders will be equal.

36. If equal quantities be multiplied by the same, or by equal quantities, the products will be equal.

37. If equal quantities be divided by the same or equal quantities, the quotients will be equal.

38. If the same quantity be added to and subtracted from another, the value of the other will not be altered.

39. If a quantity be both multiplied and divided by another, its value will not be altered.

40. If one quantity be greater than another, and the same or equal quantities be added to, or taken from both; or if both be multiplied or divided by the same or equal quantities, the results of the several operations on the greater will be greater than those of the same operations on the less.

ADDITION.

41. By Addition in Algebra is meant the connecting of the several expressions by their proper signs, and the collecting of like quantities into one term whose coefficient is the difference, with its proper sign, of the sums of the coefficients of the positive and negative terms respectively.

In performing the operation, the quantities to be added are either written in a continued line, or placed under each other, as in Arithmetic; those quantities which are similar or like, are then collected into one term, and the result is written in a line.

EXAMPLES.

(1)
$$\begin{array}{rrr} 2a & +\ \ b & -\ 3x \\ 3a & -\ 2b & +\ \ x \\ a & +\ \ b & -\ 5x \\ \hline 6a & & -\ 7x \end{array}$$

In the first column of this example the coefficients of the like quantities have the same sign, and therefore their arithmetical sum is taken, with its proper sign In the second and third columns, different signs being involved, the difference of the sums of the coefficients of the positive and negative terms is taken with its proper sign, this difference being 0 in the second column.

This example might also have been written in a line; thus

$$2a+3a+a+b-2b+b-3x+x-5x=6a+2b-2b-8x+x$$
$$=6a-7x.$$

The terms might have been arranged in any other order, but generally it is most convenient to place them alphabetically.

(2)
$$\begin{array}{r} a+b \\ a-b \\ \hline 2a \end{array}$$

(3)
$$\begin{array}{r} a^2+2ab+b^2 \\ a^2-2ab+b^2 \\ \hline 2a^2+2b^2 \end{array}$$

$$(4)\qquad \begin{array}{l} \frac{a^2}{2}+\frac{ax}{2}-\frac{x^2}{5} \\ \frac{a^2}{3}-\frac{ax}{4}+\frac{x^2}{7} \\ \hline \frac{5a^2}{6}+\frac{ax}{4}-\frac{2x^2}{35} \end{array}$$

EXAMPLES FOR PRACTICE.

(1) Required the sum of

$$a^3-3a^2b+3ab^2-b^3 \text{ and } a^3+3a^2b+3ab^2+b^3.$$

Ans. $2a^3+6ab^2$.

(2) Required the sum of

$$a^m-b^n+3x^p,\ 2a^m-3b^n-x^p \text{ and } a^m+4b^n-x^q.$$

Ans. $4a^m+2x^p-x^q$.

(3) Required the sum of

$$a-b-c,\ b+c-d,\ d-e+f,\ e-f-g.$$

Ans. $a-g$.

(4) Required the sum of

$$a^3-\frac{3a^2x}{4}+\frac{ax^2}{3}-\frac{5x^3}{8} \text{ and } a^3+\frac{a^2x}{2}+\frac{ax^2}{4}+\frac{2x^3}{3}.$$

Ans. $2a^3-\frac{a^2x}{4}+\frac{7ax^2}{12}+\frac{x^3}{24}$.

(5) Required the sum of

$$x^2-3xy-\frac{2}{3}y^2,\ 2y^2-\frac{2}{3}y^3+z^2,\ xy-\frac{1}{3}y^2+y^3,\ 2xy-\frac{1}{3}y^3.$$

Ans. $x^2+y^2+z^2$.

SUBTRACTION.

42. The operation of subtraction is denoted by the sign —, which, when combined with the signs of each of the quantities to be subtracted, changes them all: thus, if $5-2$ were subtracted from 9, the remainder would evidently be greater by 2 than if 5 only were subtracted, or

$$9-(5-2)=9-5+2:$$

similarly $a-(b-c)=a-b+c$;

which may be proved thus,

$$a-(b-c)=a+c-(b-c+c)=a+c-b \text{ or } a-b+c.$$

Hence, if the sign — be prefixed to any number of quantities included between brackets, when the brackets are removed, the sign of every term so included must be changed.

EXAMPLES.

(1) If from $2x+3y$ we take $x-2y$, we shall have

$$2x+3y-(x-2y)=2x+3y-x+2y=x+5y:$$

or placing the quantities to be subtracted under those which form the subtrahend, supposing their signs to be changed, and proceeding as in addition,

$$\begin{array}{r} 2x+3y \\ x-2y \\ \hline x+5y \end{array}$$

(2) $\begin{array}{r} a+b \\ a-b \\ \hline 2b \end{array}$ (3) $\begin{array}{r} a-b \\ a+b \\ \hline -2b \end{array}$

(4)
$$\begin{array}{r} a - b + c - d \\ a + b - c + d \\ \hline -2b + 2c - 2d \\ \hline \end{array}$$

(5)
$$\begin{array}{r} 3a - 3b + c \\ 2a + 3b - 5c \\ \hline a - 6b + 6c \\ \hline \end{array}$$

(6)
$$\begin{array}{r} 3ax + by - 3cz \\ 2ax + 2by + 5cz \\ \hline ax - by - 8cz \\ \hline \end{array}$$

(7)
$$\begin{array}{r} a - \frac{3c}{4} \\ b + \frac{c}{4} \\ \hline a - b - c \\ \hline \end{array}$$

(8)
$$\begin{array}{r} \frac{5a}{2} - 7b + \frac{c}{2} - 3bc \\ \frac{3a}{2} + 3b + \frac{c}{4} + ab \\ \hline a - 10b + \frac{3c}{4} - ab - 3bc \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

(1) From $a^2+2ab+b^2$ take $a^2-2ab+b^2$.

Ans. $4ab$.

(2) $5x-4y-\{7x-4y-(3x-y)\}=x-y$.

(3) $3x^2+3y^2-\{x^2+2xy+y^2-(2xy-x^2-y^2)\}$
$=x^2+y^2$.

MULTIPLICATION.

43. The manner in which the multiplication of simple algebraical quantities is indicated is explained in Art. 9; and it may be further observed, that the order in which the quantities are placed, is a matter of indifference as regards the value of the product; for $a\times b$ or ab is equivalent to $b\times a$ or ba.

44. The sign of a product may always be determined by the following general rule.

The product of two algebraical quantities is positive when they have the same sign, and negative when they have different signs.

1st. $(+a)(+b)=+ab$; for, as in Arithmetic, $a\times b$ signifies that a is to be taken positively b times, or that b is to be taken positively a times: the product ab is therefore positive.

2nd. $(-a)(+b)=-ab$; because $-a$ is to be taken b times, or, which is the same thing, b is to be subtracted a times.

Hence also $(+a)(-b)=-ab$.

3rd. $(-a)(-b)=+ab$. Here $-a$ is to be subtracted b times, or $-b$ is to be subtracted a times; and since, by Art. 24, to subtract a negative quantity is the same as to add a positive one, the product will be $+ab$.

The 2nd and 3rd cases may be proved thus:

$$a-a=0, \text{ multiply both sides by } +b,$$

$$\text{and } ab-a(+b)=0=ab-ab,$$

$$\therefore (-a)(+b)=-ab.$$

$$\text{Again, } a-a=0, \text{ multiply both sides by } -b,$$

$$\text{and } -ab+(-a)(-b)=0=ab-ab,$$

$$\therefore (-a)(-b)=+ab.$$

Hence also the product of an odd number of negative quantities is negative, and the product of an even number of negative quantities positive.

45. When the same symbol is involved any number of times, the sum of the indices of that symbol is taken: thus,

$a^2 \times a^3$ is identical with *aaaaa*, or a^5, (Art. 18); and generally, if m and n represent any positive whole numbers, then

$$a^m a^n = a^{m+n};$$

$$\begin{aligned} \text{for, } a^m a^n &= (aaa \ldots \text{to } m \text{ factors}) \times (aaa \ldots \text{to } n \text{ factors}) \\ &= aaa \ldots \text{to } m+n \text{ factors} \\ &= a^{m+n}. \end{aligned}$$

46. If the quantities have numerical coefficients, these are multiplied as in Arithmetic, the sign and the literal product being determined by the preceding articles. Thus

$$4a \times 5b \times 3c = 60abc.$$

47. Whatever be the number of terms in the multiplier or multiplicand, every term of the latter must be multiplied by each of the former, and the sum of all the products taken for the whole product of the two quantities; for if an aggregate be multiplied by any quantity, the product is equal to the sum of the products obtained by multiplying its component parts by that quantity.

EXAMPLES.

(1) Multiply $a+b$ by $c+d$.

$$\begin{array}{c} a+b \\ c+d \\ \hline ac+bc+ad+bd \\ \hline \end{array}$$

Here $a+b$ is to be taken $c+d$ times, and $a+b$ taken c times gives $ac+bc$, and taken d times gives $ad+bd$, whence

$$(a+b)(c+d) = ac+bc+ad+bd.$$

(2) Multiply $a+b$ by $c-d$.

$$\begin{array}{c} a+b \\ c-d \\ \hline ac+bc-ad-bd \\ \hline \end{array}$$

Here $a+b$ is to be taken $c-d$ times; that is, $a+b$ is to be added c times and subtracted d times.

(3) Required the square of $a+b$.

$$\begin{array}{r}
a+b \\
a+b \\
\hline
a^2+ab \qquad \\
ab+b^2 \\
\hline
a^2+2ab+b^2
\end{array}$$

(4) Multiply $a+b$ by $a-b$.

$$\begin{array}{r}
a+b \qquad \\
a-b \qquad \\
\hline
a^2+ab \qquad \\
-ab-b^2 \\
\hline
a^2-b^2
\end{array}$$

Hence it appears, that *the product of the sum and difference of two quantities is equal to the difference of their squares*, which is proved in Prop. 5, Book II. of Euclid's Elements, and furnishes a most useful rule for the transformation and reduction of algebraical expressions.

(5) Find the cube of $x+a$.

$$\begin{array}{l}
x+a \\
x+a \\
\hline
x^2+ax \\
\qquad ax+a^2 \\
\hline
x^2+2ax+a^2=(x+a)^2 \\
x+a \\
\hline
x^3+2ax^2+a^2x \\
\qquad ax^2+2a^2x+a^3 \\
\hline
x^3+3ax^2+3a^2x+a^3=(x+a)^3
\end{array}$$

(6) Multiply $x+a$ by $x+b$.

$$\begin{array}{l} x+a \\ x+b \\ \hline x^2+ax \\ \quad\;\; bx+ab \\ \hline x^2+ax+bx+ab, \text{ or } x^2+(a+b)x+ab. \end{array}$$

This result is arranged according to powers of x; ax and bx being considered as like quantities and collected into one term.

(7) Multiply $\frac{3}{4}a^2+\frac{5}{6}ax+x^2$ by $\frac{3}{4}a^2-\frac{5}{6}ax+x^2$.

$$\begin{array}{llllll} \frac{3}{4}a^2 & +\frac{5}{6}ax & +x^2 & & & \\ \frac{3}{4}a^2 & -\frac{5}{6}ax & +x^2 & & & \\ \hline \frac{9}{16}a^4 & +\frac{5}{8}a^3x & +\frac{3}{4}a^2x^2 & & & \\ & -\frac{5}{8}a^3x & -\frac{25}{36}a^2x^2 & -\frac{5}{6}ax^3 & & \\ & & \frac{3}{4}a^2x^2 & +\frac{5}{6}ax^3 & +x^4 & \\ \hline \frac{9}{16}a^4 & & +\frac{29}{36}a^2x^2 & & x^4 & \end{array}$$

EXAMPLES FOR PRACTICE.

(1) Required the product of $3a$, $4ab$, and $5abc$.

Ans. $60a^3b^2c$.

(2) Required the square of $a-b$.

Ans. $a^2-2ab+b^2$.

(3) Multiply x^2-ax+a^2 by x^2+ax+a^2.

Ans. $x^4+a^2x^2+a^4$.

(4) Required the cube of $a-x$.

Ans. $a^3-3a^2x+3ax^2-x^3$.

(5) Multiply x^2-x+1 by x^2+x-1.

Ans. x^4-x^2+2x-1.

(6) Multiply $x^4-x^3+x^2-x+1$ by x^2+x-1.

Ans. $x^6-x^4+x^3-x^2+2x-1$.

(7) Multiply $1-x+x^2-x^3$ by $1+x$.

Ans. $1-x^4$.

(8) Required the product of $x+2$, $x+4$, and $x+6$.

Ans. $x^3+12x^2+44x+48$.

(9) Multiply $a+b+c$ by $a+b-c$.

Ans. $a^2+2ab+b^2-c^2$.

(10) Required the product of $x-a$ and $x-b$.

Ans. $x^2-(a+b)x+ab$.

(11) Multiply x^3-3x^2+3x-1 by x^2+3x+1.

Ans. $x^5-5x^3+5x^2-1$.

(12) Multiply $a^4-a^2b^2+b^4$ by a^2-b^2.

Ans. $a^6-2a^4b^2+2a^2b^4-b^6$.

(13) Multiply $a^2-2ax+4x^2$ by $a^2+2ax+4x^2$.

Ans. $a^4+4a^2x^2+16x^4$.

(14) Multiply $a^3+3a^2x+3ax^2+x^3$ by $a^3-3a^2x+3ax^2-x^3$.

Ans. $a^6-3a^4x^2+3a^2x^4-x^6$.

(15) Multiply $x^2-\frac{1}{2}ax+a^2$ by $x^2+\frac{1}{2}ax-a^2$.

Ans. $x^4-\frac{1}{4}a^2x^2+a^3x-a^4$.

(16) Multiply $a^3+3a^2b-2ab^2+3b^3$ by $a^2+2ab-3b^2$.

Ans. $a^5+5a^4b+a^3b^2-10a^2b^3+12ab^4-9b^5$.

(17) Multiply $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$ by $x+y$.

Ans. x^6-y^6.

(18) Multiply $(x+y)^2+2y^2$ by $(x-y)^2$.

Ans. $x^4-4xy^3+3y^4$.

DIVISION.

48. The rules for Division are, in all cases, deduced from the consideration of its being the inverse of Multiplication.

49. If the divisor and dividend be affected with like signs, the sign of the quotient is $+$: but if their signs be unlike, the sign of the quotient is $-$.

This follows immediately from the inverse operation: thus

$$\frac{+ab}{+a}=+b, \quad \because (+a)(+b)=+ab;$$

$$\frac{-ab}{-a}=+b, \quad \because (-a)(+b)=-ab;$$

$$\frac{-ab}{+a}=-b, \quad \because (+a)(-b)=-ab;$$

and $$\frac{+ab}{-a}=-b, \quad \because (-a)(-b)=+ab.$$

50. In the division of monomials, whenever the entire divisor is found in the dividend, the other part of the dividend, with the sign determined as above, is the quotient: thus

$\dfrac{abcx}{ab}=cx$; because ab multiplied by cx gives $abcx$.

The result will be the same if we divide first by a and then by b; for $\frac{abcx}{a} = bcx$, and $\frac{bcx}{b} = cx$ as before.

51. Similarly, $\frac{aaaaa}{aaa} = aa$, or $\frac{a^5}{a^3} = a^2$; and, generally, since $a^m = aaa\ldots$ to m factors, and $a^n = aaa\ldots$ to n factors,

$$\frac{a^m}{a^n} = \frac{aaa \ldots \text{ to } m \text{ factors}}{aaa \ldots \text{ to } n \text{ factors}};$$

which, by rejecting the same factors from the dividend and divisor, $= aaa\ldots$to $(m-n)$ factors, if m be greater than n,

$$= a^{m-n};$$

but which $= \frac{1}{aaa \ldots (n-m) \text{ factors}}$, if m be less than n,

$$= \frac{1}{a^{n-m}}.$$

If, therefore, we are not supposed to know whether m be greater or less than n, we must regard

$$a^{m-n} \text{ and } \frac{1}{a^{n-m}} \text{ or } \frac{1}{a^{-(m-n)}} \text{ (Art. 42.)}$$

as equivalent expressions: and hence it appears that a quantity in the divisor, or the denominator of a fraction, may be transferred to the dividend or to the numerator of a fraction, and vice versâ, if the sign of its index be changed. Thus,

$$\frac{1}{a} = a^{-1}, \quad abc^{-n}d^{-n} = \frac{ab}{c^n d^n}, \quad \frac{1}{(a+x)^n} = (a+x)^{-n},$$

$$a + x = \frac{1}{(a+x)^{-1}}.$$

Hence also $\frac{a^m}{a^m} = a^{m-m} = a^0$,

but $\frac{a^m}{a^m} = 1$,

and $\therefore a^0 = 1$.

It appears, therefore, that the corresponding terms of the two following series, viz.

$$\ldots a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ \ldots$$

and $$\ldots a^3,\ a^2,\ a,\ 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \ldots$$

are equivalent expressions, each term of the former being obtained from the preceding by subtracting 1 from the index of the symbol, and each term of the latter by dividing the preceding term by the symbol itself.

52. If only part of the divisor be contained in the dividend the quotient is represented fractionally, the part which is common to the divisor and dividend being cancelled: thus

$$a^3b^2c \text{ divided by } -a^2bx, \text{ or } \frac{a^3b^2c}{-a^2bx} = -\frac{abc}{x}.$$

If a^3b^2c be first divided by $-a^2b$ the quotient is $-abc$, which quantity remains to be divided by x, but not being a multiple of x, the division is represented by $-\frac{abc}{x}$.

53. If the dividend consist of several terms, and the divisor be a monomial, every term of the dividend must be divided by it: thus

$$\frac{ax^3 - 3abx^4 + 6acx^5}{ax^2} = x - 3bx^2 + 6cx^3.$$

54. When the divisor also consists of several terms, arrange both the divisor and dividend, if possible, according to the powers of some letter common to both, and place them in one line, as in Arithmetic: find the quantity which multiplied into the first term of the divisor will produce the first term of the dividend; this is the first term of the quotient: multiply this term into the divisor, and subtract the result from the dividend; consider the remainder, if any, as a new dividend, and proceed as before.

By this process the product of the divisor and all the terms of the quotient is subtracted from the dividend: and when there is no remainder the complete quotient is obtained, but when there is a remainder the division is interminable.

The value of a remainder is represented by a fraction of which it is the numerator, and the divisor the denominator; thus expressed it may, with its proper sign, be annexed to the quotient.

EXAMPLES.

(1) Divide $a^2-2ab+b^2$ by $a-b$.

$$\begin{array}{r|l} a-b) & a^2-2ab+b^2\ (a-b \\ & \underline{a^2-\ ab\quad\ \ } \\ & -\ ab+b^2 \\ & \underline{-\ ab+b^2} \\ & \cdot\quad\cdot \end{array}$$

The reason of the process is simply this, that the divisor is contained in the whole dividend as often as it is contained in all its parts.

In this example we inquire first how often a is contained in a^2, which gives a for the first term of the quotient; then multiplying the whole divisor by it, we have $a^2 - ab$ to be subtracted from the dividend, and the remainder is $-ab + b^2$, with which we are to proceed as before. The whole quantity $a^2 - 2ab + b^2$ is thus resolved into two parts, each of which is divided by $a-b$: whence the true quotient is obtained.

(2) Divide $x^4-4x^3+6x^2-4x+1$ by x^2-2x+1.

$$\begin{array}{l}
x^2-2x+1)\ x^4-4x^3+6x^2-4x+1\ (x^2-2x+1 \\
\qquad\qquad\quad \underline{x^4-2x^3+\ x^2} \\
\qquad\qquad\qquad -2x^3+5x^2-4x \\
\qquad\qquad\qquad \underline{-2x^3+4x^2-2x} \\
\qquad\qquad\qquad\qquad\quad x^2-2x+1 \\
\qquad\qquad\qquad\qquad\quad \underline{x^2-2x+1} \\
\qquad\qquad\qquad\qquad\quad \cdot \quad \cdot \quad \cdot
\end{array}$$

(3) Divide $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $\frac{4}{3}x-2$.

$$\begin{array}{l}
\frac{4}{3}x-2\Big)\ x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2\ \Big(\frac{3}{4}x^3-\frac{1}{2}x^2+1 \\
\qquad\quad \underline{x^4-\frac{3}{2}x^3} \\
\qquad\qquad -\frac{2}{3}x^3+x^2 \\
\qquad\qquad \underline{-\frac{2}{3}x^3+x^2} \\
\qquad\qquad\qquad\qquad \frac{4}{3}x-2 \\
\qquad\qquad\qquad\qquad \underline{\frac{4}{3}x-2} \\
\qquad\qquad\qquad\qquad \cdot \qquad \cdot
\end{array}$$

(4) Divide $a^8+16a^4b^4+256b^8$ by $a^4-4a^2b^2+16b^4$.

$$a^4-4a^2b^2+16b^4)\ a^8+16a^4b^4+256b^8\ (a^4+4a^2b^2+16b^4$$
$$\underline{a^8-\ 4a^6b^2+16a^4b^4}$$
$$4a^6b^2$$
$$\underline{4a^6b^2-16a^4b^4+64a^2b^6}$$
$$16a^4b^4-64a^2b^6+256b^8$$
$$\underline{16a^4b^4-64a^2b^6+256b^8}$$
$$\cdot\quad\cdot\quad\cdot$$

(5) Divide $mpx^3-(mq-np)\,x^2+(mr-nq)\,x+nr$ by $mx+n$.

$$mx+n)\ mpx^3-(mq-np)x^2+(mr-nq)\,x+nr\ (px^2-qx+r$$
$$\underline{mpx^3\qquad +npx^2}$$
$$-mqx^2+(mr-nq)\,x$$
$$\underline{-mqx^2\qquad -nqx}$$
$$mrx+nr$$
$$\underline{mrx+nr}$$
$$\cdot\quad\cdot$$

(6) Divide $x^3-2ax^2+(a^2-ab-b^2)\,x+a^2b-ab^2$ by $x^2-(a+b)\,x-ab$.

$$x^2-(a+b)x-ab)\ x^3-2ax^2+(a^2-ab-b^2)x+a^2b-ab^2\ (x-a+b$$
$$\underline{x^3-(a+b)\,x^2-abx}$$
$$-(a-b)\,x^2+(a^2-b^2)\,x+a^2b-ab^2$$
$$\underline{-(a-b)\,x^2+(a^2-b^2)\,x+a^2b-ab^2}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

If there be any quantity which is common to every term of the dividend and divisor, it is generally convenient to divide first by that quantity and then by the other part of the divisor.

(7) Divide $x^4-px^3+qx^2-rx$ by x^2-ax.

In this example every term of the dividend is divisible by x, which is also a factor of the divisor x^2-ax, or $(x-a)x$: if therefore we divide first by this quantity, we have

$$\frac{x^4-px^3+qx^2-rx}{x}=x^3-px^2+qx-r;$$

which remains to be divided by $x-a$ as follows,

$$\begin{array}{l}
x-a)\ x^3-px^2+qx-r\ (x^2+(a-p)x+a^2-pa+q \\
\qquad \underline{x^3-ax^2} \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{a^3-pa^2+qa-r}{x-a} \\
\qquad\qquad (a-p)x^2+qx \\
\qquad\qquad \underline{(a-p)x^2-(a^2-pa)x} \\
\qquad\qquad\qquad\qquad (a^2-pa+q)x-r \\
\qquad\qquad\qquad\qquad \underline{(a^2-pa+q)x-a(a^2-pa+q)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{\underline{a^3-pa^2+qa-r}}
\end{array}$$

(8) Divide 1 by $1+x$.

$$\begin{array}{l}
1+x)\ 1 \quad (1-x+x^2-\ldots\ldots \\
\qquad\ \ \underline{1+x} \\
\qquad\ \ -x \\
\qquad\ \ \underline{-x-x^2} \\
\qquad\qquad\quad x^2 \\
\qquad\qquad\quad \underline{x^2+x^3} \\
\qquad\qquad\quad -x^3
\end{array}$$

Three terms are sufficient to ascertain the law of the formation of the infinite series of terms in the quotient, which have alternately the signs + and −, the index of x increasing by 1 in each successive term.

(9) Divide x^4-a^4 by $x-a$.

$$\begin{array}{l}
x-a)\ x^4-a^4\ (x^3+ax^2+a^2x+a^3 \\
\quad\ \ \underline{x^4-ax^3} \\
\qquad\quad ax^3-a^4 \\
\qquad\quad \underline{ax^3-a^2x^2} \\
\qquad\qquad\quad a^2x^2-a^4 \\
\qquad\qquad\quad \underline{a^2x^2-a^3x} \\
\qquad\qquad\qquad\quad a^3x-a^4 \\
\qquad\qquad\qquad\quad \underline{a^3x-a^4} \\
\qquad\qquad\qquad\qquad \cdot\quad\cdot
\end{array}$$

(10) To shew that x^n-y^n is divisible by $x-y$, without a remainder.

$$\begin{array}{l}
x-y)\ x^n-y^n\ (x^{n-1}+x^{n-2}y+x^{n-3}y^2+\&c.+xy^{n-2}+y^{n-1} \\
\qquad\ \underline{x^n-x^{n-1}y} \\
\qquad\qquad x^{n-1}y-y^n \\
\qquad\qquad \underline{x^{n-1}y-x^{n-2}y^2} \\
\qquad\qquad\qquad x^{n-2}y^2-y^n \\
\qquad\qquad\qquad \underline{x^{n-2}y^2-x^{n-3}y^3} \\
\qquad\qquad\qquad\qquad x^{n-3}y^3-y^n
\end{array}$$

And here it may be observed, that in the remainders the index of x is diminished and that of y increased by 1 at each succeeding step, and that the sum of the indices in every term is n; so that we shall at length have $x^2y^{n-2}-y^n$ for a remainder; and continuing the division from this point, we have

$$\begin{array}{r} x^2y^{n-2}-y^n \\ \underline{x^2y^{n-2}-xy^{n-1}} \\ xy^{n-1}-y^n \\ \underline{xy^{n-1}-y^n} \\ \cdot \quad \cdot \end{array}$$

(11) To find in what case x^n-y^n is divisible by $x+y$ without a remainder, n being a positive integer.

$$x+y)\ x^n-y^n\ (x^{n-1}-x^{n-2}y+x^{n-3}y^2-\&c.$$

$$\begin{array}{l} \underline{x^n+x^{n-1}y} \\ \quad -x^{n-1}y-y^n \\ \quad \underline{-x^{n-1}y-x^{n-2}y^2} \\ \qquad\qquad x^{n-2}y^2-y^n \\ \qquad\qquad \underline{x^{n-2}y^2+x^{n-3}y^3} \\ \qquad\qquad\qquad -x^{n-3}y^3-y^n \end{array}$$

Here it is observable that in the successive remainders the index of x is diminished and that of y increased by 1 at each step, and that the sum of the indices in each term is n: so that the n^{th} remainder will be $\pm x^{n-n}y^n-y^n$, or $\pm y^n-y^n$, the sign $+$ or $-$ being used according as n is even or odd: and as this remainder cannot vanish unless its first term be positive, it follows that x^n-y^n is exactly divisible by $x+y$ only when n is even.

In the same manner it may be shewn that x^n+y^n is divisible by $x+y$ only when n is odd; and that the division of x^n+y^n by $x-y$ is interminable.

The same will be found to be true in each of these cases if n be the numerator of a fractional index whose denominator is the same in both the dividend and divisor.

EXAMPLES FOR PRACTICE.

(1) Divide x^3-5x^2-x+14 by x^2-3x-7.

Ans. $x-2$.

(2) Divide $x^4-4x^3+6x^2-4x+1$ by x^2-2x+1.

Ans. x^2-2x+1.

(3) Divide $x^4-4x^2+8x+16$ by $x+2$.

Ans. x^3-2x^2+8.

(4) Divide $2x^4-32$ by $x-2$.

Ans. $2x^3+4x^2+8x+16$.

(5) Divide x^4-81y^4 by $x-3y$.

Ans. $x^3+3x^2y+9xy^2+27y^3$.

(6) Divide x^4-y^4 by $x-y$.

Ans. $x^3+x^2y+xy^2+y^3$.

(7) Divide x^5+1 by $x+1$.

Ans. $x^4-x^3+x^2-x+1$.

(8) Divide x^6+y^6 by x^2+y^2.

Ans. $x^4-x^2y^2+y^4$.

(9) Divide $ac-ad+bc-bd$ by $a+b$.

Ans. $c-d$.

(10) Divide $x^4+x^2y^2+y^4$ by x^2-xy+y^2.

Ans. x^2+xy+y^2.

(11) Divide $x^4-2a^2x^2+16a^3x-15a^4$ by $x^2+2ax-3a^2$.

Ans. $x^2-2ax+5a^2$.

(12) Divide $x^3-86x-140$ by $x-10$.

Ans. $x^2+10x+14$.

(13) Divide $x^5-a^2x^3-a^3x^2+a^5$ by x^2-a^2.

Ans. x^3-a^3.

(14) Divide

$x^5 - 6a^2x^3 + 5a^3x^2 + 2a^4x - 2a^5$ by $x^3 - 2ax^2 + a^3$.

Ans. $x^2 + 2ax - 2a^2$.

(15) Divide $256x^4 + 16x^2y^2 + y^4$ by $16x^2 + 4xy + y^2$.

Ans. $16x^2 - 4xy + y^2$.

(16) Divide $\frac{3}{2}x^3 - \frac{5}{4}x^2 - 8x + 9$ by $3x^2 + \frac{7}{2}x - 9$.

Ans. $\frac{1}{2}x - 1$.

(17) Divide $3a^2 + 8ab + 4b^2 + 10ac + 8bc + 3c^2$ by $a + 2b + 3c$.

Ans. $3a + 2b + c$.

(18) Divide $50x^3y^2 - 6y^5 + 25xy^4 - 45x^2y^3 - 41x^4y + 20x^5$ by $5xy^2 - 3y^3 - 4x^2y + 5x^3$.

Ans. $4x^2 - 5xy + 2y^2$.

(19) Divide $x^3 + px^2 + qx + r$ by $x + a$.

Ans. $x^2 \quad (a-p)x + a^2 - ap + q - \dfrac{a^3 - a^2p + aq - r}{x+a}$.

(20) Divide $x^6 - (a-b)x^3 \;\; ab$ by $x^3 + b$.

Ans. $x^3 - a$.

(21) Divide $x^{m+n} + x^ny^n + x^my^m + y^{m+n}$ by $x^m + y^n$.

Ans. $x^n + y^m$.

(22) Divide 1 by $1 + x$ to five terms.

Ans. $1 - x + x^2 - x^3 + x^4 \ldots\ldots$

(23) $\dfrac{1+3x}{1-2x} = 1 + 5x + 10x^2 + 20x^3 +$ &c. *in inf.*

(24) $\dfrac{a}{1-x} = a + ax + ax^2 +$ &c. *in inf.*

(25) Divide $x^n + y^n$ by $x + y$, exhibiting the first three and the last three terms of the quotient, n being an odd number.

Ans. $x^{n-1} - x^{n-2}y + x^{n-3}y^2 \ldots + x^2y^{n-3} - xy^{n-2} + y^{n-1}$.

THE GREATEST COMMON MEASURE.

55. When the common factor is a monomial it is discoverable by inspection: thus, $3b$ is obviously a common factor of $6ab$ and $15b^3$, the respective quotients being $2a$ and $5b^2$; and since these quotients do not contain a common factor, $3b$ is the greatest common factor of $6ab$ and $15b^3$.

In many other instances also a common factor may be discovered by inspection: thus, the greatest common factor of $a^2+2ab+b^2$ and $ab+b^2$ is $a+b$, which is obvious when the quantities are put under the form $(a+b)^2$ and $(a+b)\,b$, the quotients being $a+b$ and b respectively.

56. When the greatest common measure cannot easily be discovered by inspection, or by trial, it may be found by the method given in the following article: but it is necessary to the investigation of this method that we should premise the following Lemmas.

LEM. 1. *If one quantity measure another, it will measure also any multiple of the other.*

For, if m measure x by the quotient p, it will measure px by the quotient pq.

LEM. 2. *If a quantity measure two others, it will measure their sum or difference.*

For, if m measure x and y by the quotients p and q, so that $pm=x$, and $qm=y$,

$$\text{then } x\pm y=pm\pm qm=(p\pm q)\,m,$$

$$\therefore m \text{ will measure } x\pm y \text{ by the quotient } p\pm q.$$

57. *To investigate a method of finding the greatest common measure of any two numbers, or the compound factor of the highest dimensions which is common to two compound algebraical expressions involving different powers of the same symbol.*

Let a and b be the given quantities, if algebraical, arranged according to the powers of the same symbol, and let

$a \div b$ give quotient p and remainder c,
$b \div c$ . . . . q . . . . d,
$c \div d$ . . . . r, and no remainders.

The operation being thus:

$$b)\,a\,(p$$
$$c)\,\underline{b}\,(q$$
$$d)\,\underline{c}\,(r$$

then d is the factor required.

For $\because c = a - pb$, every com. mea. of a and b measures c,
and $\because d = b - qc$, every com. mea. of b and c measures d,
therefore every common measure of a and b measures d.

But d measures c, by supposition,

it $\therefore$ measures $qc + d$ or b,
and $\therefore$ $pb + c$ or a;

and it has been shewn that every common measure of a and b measures d; d is therefore the greatest common measure, or the factor of the highest dimensions of a and b.

It is obvious that the proof would have been the same, had it been necessary to continue the operation through four or more stages.

Obs. 1. If the first term of any dividend be not divisible by the first term of the divisor, it may be made so by multiplying the whole dividend by any quantity which is prime to the divisor, for a common factor will not be introduced thereby.

Obs. 2. But if, in the algebraical operation, any remainder contain a factor which is prime to the divisor, it must be rejected, that the divisor may not be multiplied by this factor, and a common factor be thus introduced.

The rejection from a remainder of a factor which is prime to the corresponding divisor, will serve to shorten the operation in an arithmetical example, but is not necessary to its success, because the first observation does not apply to an arithmetical example.

58. Since every other common measure of a and b is a measure of their greatest common measure d, conversely every quantity which measures d measures a and b; now the greatest quantity which divides d is $\frac{1}{2}d$, the next in magnitude is $\frac{1}{3}d$, and so on; whence all the common measures of a and b are comprised in the series d, $\frac{1}{2}d$, $\frac{1}{3}d$, $\frac{1}{4}d$, &c.

59. If d measures any quantity, $-d$ also measures it; for -1 and d measure it, and therefore $(-1).d$, or $-d$ measures it.

EXAMPLES.

(For Arithmetical Examples, see the Author's *Arithmetic*, p. 8.)

(1) Required the compound factor of the highest dimensions which is common to x^3+2x^2+x+2 and x^4+4x^2+3.

$$
\begin{array}{l}
x^3+2x^2+x+2)\ x^4+4x^2+3\ (x-2 \\
\qquad\qquad\qquad\quad \underline{x^4+2x^3+x^2+2x} \\
\qquad\qquad\qquad\qquad -2x^3+3x^2-2x+3 \\
\qquad\qquad\qquad\qquad \underline{-2x^3-4x^2-2x-4} \\
\qquad\qquad\qquad\qquad\quad 7x^2+7=7\,(x^2+1)
\end{array}
$$

Rejecting the factor 7 according to *Obs.* 2, and proceeding with the remaining factor,

$$
\begin{array}{l}
x^2+1)\ x^3+2x^2+x+2\ (x+2 \\
\qquad\quad \underline{x^3+x} \\
\qquad\qquad 2x^2+2 \\
\qquad\qquad \underline{2x^2+2} \\
\qquad\qquad\ \cdot\quad\cdot
\end{array}
$$

whence x^2+1, being the last divisor, is the factor required.

(2) Let it be required to find the highest common factor of $a^2-5ab+4b^2$ and $a^3-a^2b+3ab^2-3b^3$.

$$
\begin{array}{l}
a^2-5ab+4b^2)\ a^3-\ a^2b+\ 3ab^2-\ 3b^3\ (a+4b \\
\qquad\qquad\qquad \underline{a^3-5a^2b+\ 4ab^2} \\
\qquad\qquad\qquad\qquad 4a^2b-\ \ ab^2-\ 3b^3 \\
\qquad\qquad\qquad\qquad \underline{4a^2b-20ab^2+16b^3} \\
\qquad\qquad\qquad\qquad\qquad 19ab^2-19b^3=19b^2\,(a-b)
\end{array}
$$

Rejecting the $19b^2$, because it is prime to the divisor, and proceeding with the remaining factor,

$$
\begin{array}{l}
a-b)\ a^2-5ab+4b^2\ (a-4b \\
\qquad\quad \underline{a^2-\ ab} \\
\qquad\qquad -4ab+4b^2 \\
\qquad\qquad \underline{-4ab+4b^2} \\
\qquad\qquad\quad \cdot\qquad\cdot
\end{array}
$$

If the terms be arranged according to the powers of b instead of a, we have

$$4b^2-5ab+a^2)\ -3b^3+3ab^2-a^2b+a^3\ ($$

Or, multiplying the dividend by -4, to make its first term divisible by the first term of the divisor,

$$\begin{array}{rl}
4b^2-5ab+a^2) & 12b^3-12ab^2+\ 4a^2b-4a^3\ (3b \\
& \underline{12b^3-15ab^2+\ 3a^2b\qquad\quad} \\
& \qquad\qquad 3ab^2+\quad a^2b-4a^3 \\
\left.\begin{array}{r}\text{or, multiplying by } 4 \\ \text{and dividing by } a,\end{array}\right\} & \qquad\quad 12\,b^2+\ 4ab-16a^2\ (3 \\
& \qquad\quad \underline{12\,b^2-15ab+\ 3a^2} \\
& \qquad\qquad\qquad 19ab-19a^2=19a\,(b-a)
\end{array}$$

$$\begin{array}{rl}
\text{and rejecting } 19a,\quad b-a) & 4b^2-5ab+a^2\ (4b-a \\
& \underline{4b^2-4ab} \\
& \quad -\ ab+a^2 \\
& \quad \underline{-\ ab+a^2} \\
& \qquad \cdot\quad\cdot
\end{array}$$

Hence, the former arrangement gives $a-b$, and the latter $b-a$ or $-(a-b)$ for the last divisor; which corresponds with what is explained in Art. 59.

(3) To find the highest common factor of $x^4+x^2y^2+y^4$ and $x^4+2x^3y+3x^2y^2+2xy^3+y^4$

$$\begin{array}{rl}
x^4+x^2y^2+y^4) & x^4+2x^3y+3x^2y^2+2xy^3+y^4\ (1 \\
& \underline{x^4\qquad\quad +\ x^2y^2\qquad\quad +y^4} \\
& \qquad 2x^3y+2x^2y^2+2xy^3=2xy\,(x^2+xy+y^2)
\end{array}$$

and rejecting $2xy$,

$$x^2+xy+y^2)\ x^4+x^2y^2+y^4\ (x^2-xy+y^2$$
$$\underline{x^4+x^3y+x^2y^2}$$
$$-x^3y+y^4$$
$$\underline{-x^3y-x^2y^2-xy^3}$$
$$x^2y^2+xy^3+y^4$$
$$\underline{x^2y^2+xy^3+y^4}$$
$$\cdot\quad\cdot\quad\cdot$$

Whence x^2+xy+y^2 is the factor required.

(4) To find the highest common factor of

$3x^2-(3c+d-3)x-3c-d$ and $2x^2+(2a+b+2)x+2a+b$.

Preparing the former of these quantities for the dividend by *Obs.* 1, Art. 57, we have

$$2x^2+(2a+b+2)x+2a+b)\ 6x^2-2(3c+d-3)x+6c-2d\ (3$$
$$\underline{6x^2+3(2a+b+2)x+6a+3b}$$
$$-(6a+3b+6c+2d)\,x-6a-3b-6c-2d$$
$$\text{or}\quad -(6a+3b-6c-2d)\,(x+1).$$

Rejecting the former factor,

$$x+1)\ 2x^2+(2a+b+2)\,x+2a+b\ (2x+2a+b$$
$$\underline{2x^2\qquad\quad +2x}$$
$$(2a+b)\,x+2a+b$$
$$\underline{(2a+b)\,x+2a+b}$$
$$\cdot\quad\cdot\quad\cdot$$

Whence $x+1$ is the factor required.

(5) To find the highest common factor of

$$2x^5-4x^4+8x^3-12x^2+6x$$
$$\text{and}\ \ 3x^5-3x^4-6x^3+9x^2-3x.$$

First

$$2x^5-4x^4+8x^3-12x^2+6x=2x\,(x^4-2x^3+4x^2-6x+3)$$

and $3x^5-3x^4-6x^3+9x^2-3x=3x(x^4-x^3-2x^2+3x-1)$.

Whence it is obvious that x is a common measure of the two quantities, but that 2 and 3 are not: the 2 and 3 may therefore be altogether rejected; the x also may be suppressed, but it must be resumed as a factor of the last divisor, in order that the result may not be affected thereby.

$$\begin{array}{r|l}
x^4-2x^3+4x^2-6x+3) & x^4-x^3-2x^2+3x-1\ (1 \\
 & \underline{x^4-2x^3+4x^2-6x+3} \\
 & \quad x^3-6x^2+9x-4
\end{array}$$

$$\begin{array}{r|l}
x^3-6x^2+9x-4) & x^4-2x^3+4x^2-6x+3\ (x+4 \\
 & \underline{x^4-6x^3+9x^2-4x} \\
 & \quad 4x^3-5x^2-2x+3 \\
 & \quad \underline{4x^3-24x^2+36x-16} \\
 & \qquad 19x^2-38x+19=19(x^2-2x+1)
\end{array}$$

$$\begin{array}{r|l}
\text{and}\quad x^2-2x+1) & x^3-6x^2+9x-4\ (x-4 \\
 & \underline{x^3-2x^2+x} \\
 & \quad -4x^2+8x-4 \\
 & \quad \underline{-4x^2+8x-4} \\
 & \qquad \cdot\quad\cdot\quad\cdot
\end{array}$$

The highest common factor of the given quantities is therefore $x\,(x^2-2x+1)$, or x^3-2x^2+x.

EXAMPLES FOR PRACTICE.

(1) To find the greatest common measure of

$$x^2+2x+1 \text{ and } x^3+2x^2+2x+1.$$

Ans. $x+1$.

(2) To find the highest common factor of

$$x^3-3x^2+7x-21 \text{ and } 2x^4+19x^2+35.$$

Ans. x^2+7.

(3) To find the highest common factor of

$$a^2+b^2+c^2+2ab+2ac+2bc \text{ and } a^2-b^2-c^2-2bc.$$

Ans. $a+b+c$.

(4) To find the factor of the highest dimensions which is common to

$$36a^6-18a^5-27a^4+9a^3 \text{ and } 27a^5b^2-18a^4b^2-9a^3b^2.$$

Ans. $9a^3(a-1)$.

(5) To find the factor of the highest dimensions which is common to

$$x^6+x^2y-x^4y^2-y^3 \text{ and } x^4-x^2y-xy^2+y^3.$$

Ans. x^2-y^2.

(6) To find the factor of the highest dimensions which is common to

$$x^6+4x^5-3x^4-16x^3+11x^2+12x-9$$
$$\text{and } 6x^5+20x^4-12x^3-48x^2+22x+12.$$

Ans. x^3+x^2-5x+3.

(7) To find the factor of the highest dimensions which is common to

$$x^4-(a^2+b^2)x^2+a^2b^2$$
$$\text{and } x^4-(a+b)^2x^2+2ab(a+b)x-a^2b^2.$$

Ans. $x^2-(a+b)x+ab$.

(8) To find the factor of the highest dimensions which is common to

$$np^3q+3np^2q^2-2npq^3-2nq^4$$
$$\text{and } 2mp^2q^2-4mp^4-mp^3q+3mpq^3.$$

Ans. $p-q$.

(9) To find the factor of the highest dimensions which is common to

$$x^4 - 2p(p-q)x^2 + (p^2+q^2)(p-q)x - p^2q^2$$

$$\text{and } x^4 - (p-q)x^3 + (p-q)q^2x - q^4.$$

Ans. $x^2 - (p-q)x + q^2$.

(10) To find the highest common factor of

$$ax^6 + (a+b)x^5 + (a+b+c)x^4 + (a+b+c+y)x^3$$
$$+ (b+c+y)x^2 + (c+y)x + y \text{ and}$$
$$ax^5 + (a+b)x^4 + (a+b+c)x^3 + (a+b+c)x^2 + (b+c)x + c.$$

Ans. $x^3 + x^2 + x + 1$.

60. When different powers of the same symbol are not involved, the given quantities may be resolved into their component factors.

Thus, let the quantities proposed be

$$ac + ad + bc + bd \text{ and } ae + af + be + bf;$$

then by forming the coefficient of each symbol separately, we obtain for them the following expressions,

$$(a+b)(c+d) \text{ and } (a+b)(e+f),$$

in which the common factor $a + b$ is manifest.

In the same manner

$$3bcq + 30mp + 18bc + 5mpq \text{ and } 4adq - 42fg + 24ad - 7fgq$$

are found to contain the common factor $q + 6$.

61. *To find the highest common factor of three or more quantities.*

Let a, b, c be any three algebraical quantities; find d the highest common factor of a and b, and then the highest

common factor of d and c: this will be the highest common factor of a, b, and c.

For every common factor of a and b is a factor of d, and therefore the highest common factor of d and c is the highest common factor of a, b, and c.

In a similar manner we may find the highest common factor of four or more quantities.

EXAMPLES FOR PRACTICE.

(1) To find the highest common factor of

$a^3+a^2b-ab^2-b^3$, $a^3-2a^2b-ab^2+2b^3$, and $a^3-3ab^2+2b^3$

Ans. $a-b$.

(2) To find the highest common factor of

$x^2-ax-2a^2$, x^2-4a^2, $x^2+ax-6a^2$, and $x^2+2ax-8a^2$.

Ans. $x-2a$.

THE LEAST COMMON MULTIPLE.

62. The least or lowest common multiple of monomials, and of many compound quantities, may be found by inspection.

The least common multiple of $2a^2b$ and $3ab^2$ is obviously $6a^2b^2$: of axy, a^2y^2, and $abxy$, is a^2bxy^2: and of $ab(x+y)$, $bc(x-y)$, and $cd(x^2-y^2)$, is $abcd(x^2-y^2)$.

63. *To find the least common multiple of any two quantities.*

Let a and b be the two quantities, d their greatest common measure, such that $a=pd$ and $b=qd$; then p and q are prime to each other, and therefore the least quantity which is divisible by each of them is pq; whence the least common

multiple of pd and qd, or of a and b, is manifestly pqd, or $pdqd \div d$, that is $ab \div d$.

Hence the least common multiple of any two quantities may be found by dividing their product by their greatest common measure.

Ex. To find the least common multiple of

$$3x^2+13x+12 \text{ and } x^2+4x+3.$$

The greatest common measure of these quantities is $x+3$; therefore their least common multiple is

$$\frac{(3x^2+13x+12)(x^2+4x+3)}{x+3}$$

$$=(3x^2+13x+12)(x+1)=3x^3+16x^2+25x+12.$$

EXAMPLES FOR PRACTICE.

(1) To find the least common multiple of

$$a^3+a^2b \text{ and } a^2-b^2.$$

Ans. $a^4-a^2b^2$.

(2) To find the least common multiple of

$$ab+ad \text{ and } ab-ad.$$

Ans. ab^2-ad^2.

(3) To find the least common multiple of

$$x^3+1 \text{ and } (x+1)^2.$$

Ans. x^4+x^3+x+1.

(4) To find the least common multiple of

$$x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$$
$$\text{and } x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5.$$

Ans. x^6-y^6.

(5) To find the least common multiple of

$x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$.

Ans. $x^6-(p+q)x^5+\{p-1+(p+1)q\}x^4-\{(p-1)p+(q-1)q\}x^3-(p+q)x^2+(p^2+q^2)x-pq$.

(6) To find the least common multiple of

$$x^4-(p^2+q^2)x^2+p^2q^2$$

and $x^4-(p+q)^2x^2+2(p+q)pqx-p^2q^2$.

Ans. $x^6+(p+q)x^5-(p^2+pq+q^2)x^4-(p+q)(p^2+q^2)x^3+(p^2+pq+q^2)pqx^2+(p+q)p^2q^2x-p^3q^3$.

64. *Every other common multiple of* a *and* b *is a multiple of their least common multiple.*

For let n be any other common multiple of the two quantities a and b, and, if possible, let m, their least common multiple, be contained in n, r times, with a remainder s, which is less than m; then $n-rm=s$, and since a and b measure n and rm, they measure $n-rm$, or s; that is, they have a common multiple less than m, which is contrary to the supposition.

65. Hence all the common multiples of a and b are comprised in the series m, $2m$, $3m$, $4m$, &c.; for the least number which m divides is $2m$, the next $3m$, and so on.

66. *To find the least common multiple of three or more quantities.*

Let a, b, and c be three given quantities: first find m the least common multiple of a and b, and then m' the least common multiple of m and c; m' is the least common multiple of a, b, and c.

For every common multiple of a and b is a multiple of m; therefore every common multiple of a, b, and c is a multiple of m and c: also every multiple of m and c is a multiple of a, b, and c; consequently the least common multiple m' of m and c is the least common multiple of a, b, and c.

The same method may be extended to four or more quantities.

EXAMPLES FOR PRACTICE.

(1) To find the least common multiple of

$$x^3-3x^2+3x-1,\quad x^3-x^2-x+1,\quad x^4-2x^3+2x-1,$$
$$\text{and } x^4-2x^3+2x^2-2x+1.$$

Ans. $x^6-2x^5+x^4-x^2+2x-1$.

(2) To find the least common multiple of

$$x+a,\quad x-a,\quad x^2-a^2, \text{ and } x^2+a^2.$$

Ans. x^4-a^4.

(3) To find the least common multiple of

$$a^2-b^2,\ a^2+b^2,\ (a-b)^2,\ (a+b)^2,\ a^3-b^3, \text{ and } a^3+b^3.$$

Ans. $a^{10}-a^6b^4-a^4b^6+b^{10}$.

FRACTIONS.

67. The operations performed on algebraical fractions are analogous to those which are performed on numerical fractions.

68. *If both the numerator and denominator of a fraction be multiplied or divided by the same quantity, the value of the fraction is not altered.*

Let any fraction $\frac{a}{b}=x$; then $\frac{a}{b}\cdot b=b\cdot x$;

or, since $\frac{a}{b}$ represents the division of a by b, $\frac{a}{b} \cdot b = a = bx$;

$\therefore ma = mbx$; or dividing by mb, $\frac{ma}{mb} = x = \frac{a}{b}$.

Hence, if both the terms of a fraction $\frac{a}{b}$ be multiplied by any quantity m its value will not be altered; and conversely, if the terms of a fraction $\frac{ma}{mb}$ be divided by m, the result will be the equivalent fraction $\frac{a}{b}$.

69. By multiplying its numerator and denominator by -1, we have the fraction $\frac{a}{b} = \frac{-a}{-b}$; which shews that if the signs of all the terms in both the numerator and denominator of a fraction be changed, its value will not be altered.

70. *To reduce a fraction to its lowest terms.*

Divide its numerator and denominator by the highest factor which is common to them; or by a common factor continually until they contain no common factor.

EXAMPLES FOR PRACTICE.

(1) $\frac{ax - x^2}{a^2 - x^2} = \frac{(a-x)x}{a^2 - x^2} = \frac{x}{a+x}$;

(See Ex. 4, p. 16.)

(2) Reduce $\frac{n^2 - 2n + 1}{n^2 - 1}$ to its lowest terms.

Ans. $\frac{n-1}{n+1}$.

(3) Reduce $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$ to its lowest terms.

Ans. $\dfrac{3x-1}{x^2-1}$.

(4) Reduce $\dfrac{x^3+2x^2+2x+1}{x^3-2x-1}$ to its lowest terms.

Ans. $\dfrac{x^2+x+1}{x^2-x-1}$.

(5) Reduce $\dfrac{a^3-2a^2}{a^2-4a+4}$ to its lowest terms.

Ans. $\dfrac{a^2}{a-2}$.

(6) Reduce $\dfrac{a^4-b^4}{a^5-a^3b^2}$ to its lowest terms.

Ans. $\dfrac{a^2+b^2}{a^3}$.

(7) Reduce $\dfrac{20x^4+x^2-1}{25x^4+5x^3-x-1}$ to its lowest terms.

Ans. $\dfrac{4x^2+1}{5x^2+x+1}$.

(8) Reduce $\dfrac{15x^3+35x^2+3x+7}{27x^4+63x^3-12x^2-28x}$ to its lowest terms.

Ans. $\dfrac{5x^2+1}{9x^3-4x}$.

(9) Reduce $\dfrac{3a^4+a^2b^2-2b^4}{2a^4+5a^2b^2+3b^4}$ to its lowest terms.

Ans. $\dfrac{3a^2-2b^2}{2a^2+3b^2}$.

(10) Reduce $\dfrac{x^3-5ax^2-16a^2x+14a^3}{x^3-7ax^2-2a^2x+14a^3}$ to its lowest terms.

Ans. $\dfrac{x^2+2ax-2a^2}{x^2-2a^2}$.

(11) Reduce $\dfrac{x^4-ax^3+a^3x-a^4}{x^3-a^3}$ to its lowest terms.

Ans. $\dfrac{x^3+a^3}{x^2+ax+a^2}$.

(12) Reduce $\dfrac{x^2+(a-b)\,x-ab}{x^2+(a+b)\,x+ab}$ to its lowest terms.

Ans. $\dfrac{x-b}{x+b}$.

(13) Reduce $\dfrac{x^2-(a-b)\,x-ab}{x^3+bx^2+ax+ab}$ to its lowest terms.

Ans. $\dfrac{x-a}{x^2+a}$.

(14) Reduce $\dfrac{x^4+(2b^2-a^2)\,x^2+b^4}{x^4+2ax^3+a^2x^2-b^4}$ to its lowest terms.

Ans. $\dfrac{x^2-ax+b^2}{x^2+ax-b^2}$.

(15) Reduce $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$ to its lowest terms.

Ans. $\dfrac{x^{m-1}}{ab+b^2x}$.

(16) Shew that $\dfrac{ac+bd+ad+bc}{af+2bx+2ax+bf}=\dfrac{c+d}{f+2x}$.

(17) Shew that

$$\frac{3x^2-(4a+2b)\,x+2ab+a^2}{x^3-(2a+b)\,x^2+(2ab+a^2)\,x-a^2b}=\frac{3x-a-2b}{(x-a)\,(x-b)}.$$

(18) Shew that

$$\frac{e^{2x}x^3+e^{2x}-x^3-1}{e^{2x}x^2+2e^xx^2-e^{2x}-2e^x+x^2-1}=\frac{(e^x-1)\,(x^2-x+1)}{(e^x+1)\,(x-1)}.$$

71. *To transform fractions having different denominators to others of equal value with a common denominator.*

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ represent any two fractions;

then $\dfrac{a}{b}=\dfrac{ad}{bd}$ and $\dfrac{c}{d}=\dfrac{bc}{bd}$, (Art. 68);

whence the proposed fractions are respectively equivalent to $\frac{ad}{bd}$ and $\frac{bc}{bd}$, which have the common denominator bd: and the process being the same whatever be the number of fractions proposed, the required transformation will be effected if each numerator be multiplied by all the denominators, except its own, for a new numerator, and all the denominators together for a new denominator.

Ex. To transform $\frac{a+1}{a-1}$, $\frac{a-1}{a+1}$, and $\frac{a^2-1}{a^2+1}$, so as to have a common denominator.

Multiplying each numerator into all the denominators, we obtain

$$(a+1)(a+1)(a^2+1)=a^4+2a^3+2a^2+2a+1,$$
$$(a-1)(a-1)(a^2+1)=a^4-2a^3+2a^2-2a+1,$$
$$\text{and } (a^2-1)(a-1)(a+1)=a^4-2a^2+1$$

for the new numerators: and multiplying all the denominators together, we have

$$(a-1)(a+1)(a^2+1)=a^4-1$$

for the common denominator: whence the required fractions are

$$\frac{a^4+2a^3+2a^2+2a+1}{a^4-1}, \quad \frac{a^4-2a^3+2a^2-2a+1}{a^4-1},$$
$$\text{and } \frac{a^4-2a^2+1}{a^4-1}.$$

72. If, however, the denominators of the proposed fractions have a common measure, and the above method be applied, the resulting fractions will admit of reduction to

lower terms, and still have a common denominator; but if the quotient arising from the division of the least common multiple of all the denominators by each denominator be multiplied by the corresponding numerator, and the least common multiple of the denominators be made the common denominator, the given fractions will evidently be at once transformed to others of equal value, having the least possible common denominator.

EXAMPLES.

(1) Let the proposed fractions be $\frac{ab}{cd}$ and $\frac{fg}{de}$.

Then, the former method gives $\frac{abde}{cd^2e}$ and $\frac{cdfg}{cd^2e}$,

which may be reduced to $\frac{abe}{cde}$ and $\frac{cfg}{cde}$;

the least common denominator cde being obviously the least common multiple of the denominators of the given fractions; whose terms are respectively multiplied by e and c; the quotients arising from the division of cde by the denominators cd and de.

(2) To transform $\frac{a}{x-1}$, $\frac{b}{x^2-1}$, and $\frac{c}{a^2-2x+1}$,

so as to have the least common denominator.

The least common multiple of the denominators being x^3-x^2-x+1, the required fractions are

$$\frac{a(x^2-1)}{x^3-x^2-x+1}, \quad \frac{b(x-1)}{x^3-x^2-x+1}, \text{ and } \frac{c(x+1)}{x^3-x^2-x+1}.$$

(3) To transform $\frac{a^2}{a^2-b^2}$ and $\frac{a^3}{a^3-b^3}$,

so as to have the least common denominator.

$$Ans.\ \frac{a^2(a^2+ab+b^2)}{a^4+a^3b-ab^3-b^4} \text{ and } \frac{a^3(a+b)}{a^4+a^3b-ab^3-b^4}.$$

ADDITION AND SUBTRACTION OF FRACTIONS.

73. To add or subtract fractions, transform them, if necessary, to equivalent fractions having a common denominator, then add or subtract the numerators as indicated, and subjoin the common denominator.

For if $\frac{a}{b}$ and $\frac{c}{d}$ be two proposed fractions,

and we suppose $\frac{a}{b}=x$ and $\frac{c}{d}=y$,

then $a=bx$ and $c=dy$,

$\therefore\ ad=bdx$ and $bc=bdy$,

$\therefore\ bdx \pm bdy$ or $bd(x \pm y)=ad \pm bc$;

and dividing each of these equals by bd, we have

$$x \pm y \text{ or } \frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}.$$

EXAMPLES FOR PRACTICE.

(1) $\frac{a+b}{2}+\frac{a-b}{2}=a.$

(2) $\frac{a+b}{2}-\frac{a-b}{2}=b,$

(3) $$\frac{x}{x+y}+\frac{y}{x-y}=\frac{x^2-xy}{x^2-y^2}+\frac{xy+y^2}{x^2-y^2}=\frac{x^2+y^2}{x^2-y^2}.$$

(4) $$\frac{1}{1-x}-\frac{1}{1+x}=\frac{1+x}{1-x^2}-\frac{1-x}{1-x^2}=\frac{2x}{1-x^2}.$$

(5) $$\frac{a+x}{a-x}+\frac{a-x}{a+x}=\frac{a^2+2ax+x^2}{a^2-x^2}+\frac{a^2-2ax+x^2}{a^2-x^2}$$

$$=\frac{2(a^2+x^2)}{a^2-x^2}.$$

(6) $$\frac{a^2+x^2}{a^2-x^2}-\frac{a-x}{a+x}=\frac{a^2+x^2}{a^2-x^2}-\frac{a^2-2ax+x^2}{a^2-x^2}=\frac{2ax}{a^2-x^2}.$$

(7) $$\frac{x}{1-x}+\frac{x^2}{(1-x)^2}=\frac{x(1-x)+x^2}{(1-x)^2}=\frac{x}{(1-x)^2}.$$

(8) $$\frac{1}{1-x}-\frac{2}{1-x^2}=\frac{1+x}{1-x^2}-\frac{2}{1-x^2}=\frac{x-1}{1-x^2};$$ and $\frac{x-1}{1-x^2}$,

or $-\frac{1-x}{1-x^2}$, reduced to its lowest terms, $=-\frac{1}{1+x}$.

(9) $$\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}$$

$$=\frac{1}{2a^2}\left(\frac{1}{a^2-x^2}+\frac{1}{a^2+x^2}\right)=\frac{1}{a^4-x^4}.$$

(10) $$\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}=\frac{85a-20b}{84}.$$

(11) $$\frac{a}{c}+\frac{(ad-bc)x}{c(c-dx)}=\frac{a-bx}{c-dx}.$$

(12) $$\frac{1}{3(1+x)}+\frac{2-x}{3(1-x+x^2)}=\frac{1}{1+x^3}.$$

(13) $\frac{a}{10}+\frac{b}{10^2}-\frac{c}{10^3}=\frac{100a+10b-c}{1000}.$

(14) $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$

$$=\frac{1+x+x^2}{1-x-x^4+x^5}.$$

(15) $\frac{2a}{(a-2x)^2}+\frac{2a+x}{(a+x)(a-2x)}-\frac{5}{a+x}=\frac{20ax-22x^2}{(a+x)(a-2x)^2}.$

(16) $\frac{a}{(a-b)(x+a)}-\frac{b}{(a-b)(x+b)}=\frac{x}{(x+a)(x+b)}.$

(17) $\frac{1}{(b-a)(c-a)(x+a)}+\frac{1}{(a-b)(c-b)(x+b)}$

$$+\frac{1}{(a-c)(b-c)(x+c)}=\frac{1}{(x+a)(x+b)(x+c)}.$$

(18) $\frac{am^2+bm+c}{(m+n)(x-m)^3}+\frac{am(m+2n)+bn-c}{(m+n)^2(x-m)^2}$

$$+\frac{an^4-bn+c}{(m+n)^3(x-m)}+\frac{an^2-bn+c}{(m+n)^3(x+n)}=\frac{ax^2+bx+c}{(x-m)^3(x+n)}.$$

74. To reduce mixed expressions to a single term, consider the integral expressions under the equivalent form of fractions whose denominators are 1, and proceed as in the foregoing Article.

(1) $2m+\frac{m}{a^m-1}+\frac{a^m m}{1-a^m}=\frac{2m(a^m-1)+m-a^m m}{a^m-1}=m.$

(2) $\frac{a^2}{a-b}-a=\frac{ab}{a-b}.$

(3) $a+x-\dfrac{4ax}{a+x}=\dfrac{(a-x)^2}{a+x}$.

(4) $1+x+\dfrac{x^2}{1-x}=\dfrac{1}{1-x}$.

(5) $1-x+x^2-\dfrac{x^3}{1+x}=\dfrac{1}{1+x}$.

(6) $1-2x+4x^2-\dfrac{6x^3-4x^4}{1+x-x^2}=\dfrac{1-x+x^2}{1+x-x^2}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

75. To *multiply* fractions, multiply the numerators for a new numerator, and the denominators for a new denominator.

For, let $\frac{a}{b}$ and $\frac{c}{d}$ be two given fractions, and suppose

$$\frac{a}{b}=x, \text{ and } \frac{c}{d}=y;$$

$$\text{then } a=bx \text{ and } c=dy,$$

$$\text{whence } ac=bd\,xy,$$

$$\text{and, dividing by } bd, \; xy \text{ or } \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

76. Hence $\left(\dfrac{a}{b}\right)^m=\dfrac{a^m}{b^m}$, and $\left(\dfrac{a}{b}\right)^{\frac{1}{m}}=\dfrac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}$.

77. *To divide one fraction by another*, invert the numerator and denominator of the divisor, and proceed as in multiplication.

For if $\frac{a}{b}$ and $\frac{c}{d}$ be supposed equal to x and y respectively,

$$\text{then } a = bx \text{ and } c = dy,$$

$$\text{also } ad = bdx \text{ and } bc = bdy;$$

whence $\frac{bdx}{bdy} = \frac{ad}{bc}$, that is, $\frac{x}{y}$ or $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$.

78. *To multiply a fraction by an integer,* multiply its numerator by the integer, or if the integer be a factor of the numerator, divide the denominator by it.

$$\text{For } \frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b} = \frac{a}{b \div c}. \text{ (Art. 68).}$$

79. *To divide a fraction by an integer,* multiply its denominator by the integer, or if the integer be a factor of the numerator, divide the numerator by it.

$$\text{For } \frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc} = \frac{a \div c}{b}.$$

EXAMPLES FOR PRACTICE.

(1) $\frac{ax}{by} \times \frac{cy}{dx} = \frac{ac}{bd}$.

(2) $\frac{5b}{a^2 - b^2} \times \frac{a + b}{15a^2} = \frac{b}{3a^2 (a - b)}$.

(3) $\frac{a^3 - b^3}{a^3 + b^3} \times \frac{(a+b)^2}{(a-b)^2} = \frac{a^2 + ab + b^2}{a^2 - ab + b^2} \times \frac{a+b}{a-b} =$

$$\frac{a^3 + 2a^2b + 2ab^2 + b^3}{a^3 - 2a^2b + 2ab^2 - b^3}.$$

(4) $\left(a - \frac{x^2}{a}\right)\left(\frac{a}{x} + \frac{x}{a}\right) = \frac{a^4 - x^4}{a^2 x}.$

(5) $\left(\frac{a}{a-b} + \frac{b}{a+b}\right) \times \left(\frac{a}{a-b} - \frac{b}{a+b}\right) =$

$$\frac{a^3(a+2b) - b^3(b-2a)}{(a^2-b^2)^2}.$$

(6) $\left(\frac{a^2}{x^2} - \frac{ab}{2xy} + \frac{b^2}{y^2}\right) \times \left(\frac{3a^2}{x^2} - \frac{2ab}{5xy} + \frac{b^2}{y^2}\right) =$

$$\frac{3a^4}{x^4} - \frac{19a^3b}{10x^3y} + \frac{21a^2b^2}{5x^2y^2} - \frac{9ab^3}{10xy^3} + \frac{b^4}{y^4}.$$

(7) $\frac{2x^2}{a^3+x^3} \div \frac{x}{x+a} = \frac{2x^2}{a^3+x^3} \times \frac{x+a}{x} = \frac{2x}{a^2 - ax + x^2}.$

(8) $\frac{x^4 - a^4}{x^2 - 2ax + a^2} \div \frac{x^2 + ax}{x-a} = \frac{x^4 - a^4}{x^2 - 2ax + a^2} \times \frac{x-a}{x(x+a)}$

$$= \frac{x^2 + a^2}{x}.$$

(9) $\frac{x^2 + 5x + 4}{x^2 + 7x + 12} \div \frac{x^2 + 2x + 1}{x^2 + 3x + 2} = \frac{x+2}{x+3}.$

(10) $\left\{x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)\right\} \div \left(x - \frac{1}{x}\right) = \left(x - \frac{1}{x}\right)^2.$

(11) $\left(\frac{a^3}{b^5} + \frac{a^4}{b^4} - \frac{7a^5}{b^3} - \frac{3a^6}{b^2} + \frac{a^2x^2}{b^2} - \frac{2a^3x^2}{b} - a^4x^2\right)$

$$\div \left(\frac{a}{b^3} + \frac{3a^2}{b^2} + x^2\right) = \frac{a^2}{b^2} - \frac{2a^3}{b} - a^4.$$

$$(12)\quad \frac{ac+(ab+bc)x+b^2x^2}{a-bx} \div \frac{a+bx}{ae+(af-be)x-bfx^2}$$
$$=ce+(cf+be)x+bfx^2.$$

80. Algebraical fractions frequently present themselves under *complex* forms, but the principles already explained are sufficient for their reduction.

EXAMPLES FOR PRACTICE.

$$(1)\quad \frac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}+\frac{g}{h}}=\frac{\frac{ad+bc}{bd}}{\frac{eh+fg}{fh}}=\frac{(ad+bc)fh}{(eh+fg)bd}.$$

$$(2)\quad \frac{\frac{a}{b}+\frac{c}{d}+\frac{e}{f}}{\frac{a'}{b'}+\frac{c'}{d'}+\frac{e'}{f'}}=\frac{(adf+bcf+bde)\,b'd'f'}{(a'd'f'+b'c'f'+b'd'e')\,bdf}.$$

$$(3)\quad \frac{1}{\frac{1}{x-a}+\frac{1}{x-b}}=\frac{x^2-(a+b)x+ab}{2x-a-b}.$$

$$(4)\quad \frac{1}{\frac{1}{x+2}+\frac{1}{x+6}+\frac{1}{x-8}}=\frac{x^3-52x-96}{3x^2-52}.$$

$$(5)\quad \frac{\frac{1}{1-x}}{\frac{1}{1-x}-1}=\frac{1}{x}.$$

$$(6)\quad \frac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}=\frac{a^2+x^2}{2ax}=\frac{1}{2}\left(\frac{a}{x}+\frac{x}{a}\right).$$

(7) $$\frac{1}{1+\frac{1}{x}} = \frac{x}{1+x}.$$

(8) $$\frac{1}{1+\frac{1}{x+\frac{1}{x}}} = \frac{x+\frac{1}{x}}{1+x+\frac{1}{x}} = \frac{1+x^2}{1+x+x^2}:$$

putting $x+\frac{1}{x}$ in the place of x in Ex. 7.

(9) $$\frac{a}{b+\frac{c}{d}} = \frac{ad}{bd+c}.$$

(10) $$\frac{a}{b+\frac{c}{d+\frac{e}{f}}} = \frac{a\left(d+\frac{e}{f}\right)}{b\left(d+\frac{e}{f}\right)+c} = \frac{adf+ae}{(bd+c)f+be}:$$

putting $d+\frac{e}{f}$ in the place of d in Ex. 9.

(11) $$\frac{x-1}{x+1+\frac{x-2}{x+2+\frac{x-3}{x+3}}} = \frac{x-1}{x+1+\frac{x^2+x-6}{x^2+6x+3}}$$

$$= \frac{x^3+5x^2-3x-3}{x^3+8x^2+10x-3}.$$

(12) $$\frac{a}{b-\frac{c}{d-\frac{e}{f-\frac{g}{h}}}} = \frac{(adf-ae)h-adg}{\{(bd-c)f-be\}h-(bd-c)g}.$$

81. Subjoined are a few examples of the conversion of fractions, by division, into infinite series involving fractional terms.

(1) In Ex. 8 of Division, we have seen that

$$\frac{1}{1+x}=1-x+x^2-\ldots.\textit{in infinitum};$$

but if we reverse the terms of the divisor, the series of terms in the quotient will be very different: thus, let it be required to convert $\frac{1}{x+1}$ into a series, by division.

$$x+1\,)\,1\,\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\&c.\right.$$

$$\begin{array}{r}
1+\frac{1}{x}\phantom{-\frac{1}{x^2}+\frac{1}{x^3}} \\
\hline
-\frac{1}{x}\phantom{-\frac{1}{x^2}+\frac{1}{x^3}} \\
-\frac{1}{x}-\frac{1}{x^2}\phantom{+\frac{1}{x^3}} \\
\hline
\frac{1}{x^2}\phantom{+\frac{1}{x^3}} \\
\frac{1}{x^2}+\frac{1}{x^3} \\
\hline
-\frac{1}{x^3}
\end{array}$$

Such indefinite quotients obviously have their origin in the rule for the formation of their several terms: for as long as there is a remainder, a factor, integral, or fractional, may be found to form a term in the quotient; and unless the divisor

be a factor of the dividend, there will necessarily always be a remainder.

$$(2)\quad \frac{a+x}{b+x}=\frac{a}{b}-\frac{(a-b)\,x}{b^2}+\frac{(a-b)\,x^2}{b^3}-\&c.$$

$$b+x\,\big)\,a+x\,\Big(\frac{a}{b}-\frac{(a-b)x}{b^2}+\frac{(a-b)x^2}{b^3}-\&c.$$

$$a+\frac{ax}{b}$$

$$x-\frac{ax}{b}=\left(1-\frac{a}{b}\right)x=\frac{(b-a)\,x}{b}=-\frac{(a-b)x}{b}$$

$$-\frac{(a-b)x}{b}-\frac{(a-b)x^2}{b^2}$$

$$\frac{(a-b)x^2}{b^2}$$

$$\frac{(a-b)x^2}{b^2}+\frac{(a-b)x^3}{b^3}$$

$$-\frac{(a-b)x^3}{b^3}$$

Also, by reversing the terms of the divisor and dividend, we obtain the following equivalent series:

$$\frac{a+x}{x+b}=\frac{a}{x}-\frac{ab}{x^2}+\frac{ab^2}{x^3}-\&c.$$

$$\frac{x+a}{b+x}=\frac{x}{b}-\frac{x^2}{b^2}+\frac{x^3}{b^3}-\&c.$$

and $\dfrac{x+a}{x+b}=1+\dfrac{a-b}{x}-\dfrac{(a-b)\,b}{x^2}+\dfrac{(a-b)\,b^2}{x^3}-\&c.$

$$(3)\quad \frac{a+x}{b-x}=\frac{a}{b}+\frac{(a+b)\,x}{b^2}+\frac{(a+b)\,x^2}{b^3}+\frac{(a+b)\,x^3}{b^4}+\&c.$$

$$(4)\quad \frac{\alpha+\beta x+\gamma x^2}{a-bx} = \frac{\alpha}{a} + \frac{kx}{a^2} + \frac{k'x^2}{a^3} + \frac{bk'x^3}{a^4} + \frac{b^2k'x^4}{a^5} + \&c.$$

$$a-bx\,\Big)\ \alpha+\beta x+\gamma x^2\ \Big(\frac{\alpha}{a} + \frac{kx}{a^2} + \frac{k'x^2}{a^3} + \frac{bk'x^3}{a^4} + \&c.$$

$$\begin{array}{l}
\alpha - \dfrac{\alpha bx}{a} \\
\hline
\dfrac{kx}{a} + \gamma x^2, \quad \text{where } k = a\beta + \alpha b \\
\dfrac{kx}{a} - \dfrac{bkx^2}{a^2} \\
\hline
\dfrac{k'x^2}{a^2}, \quad \text{where } k' = a^2\gamma + bk \\
\dfrac{k'x^2}{a^2} - \dfrac{bk'x^3}{a^3} \qquad = a^2\gamma + ab\beta + \alpha b^2 \\
\hline
\dfrac{bk'x^3}{a^3} \\
\dfrac{bk'x^3}{a^3} - \dfrac{b^2k'x^4}{a^4} \\
\hline
\dfrac{b^2k'x^4}{a^4}
\end{array}$$

$$(5)\quad \frac{a'+b'x}{a+bx} = \frac{a'}{a} - \frac{kx}{a^2} + \frac{bkx^2}{a^3} - \frac{b^2kx^3}{a^4} + \&c.$$

where $k = a'b - ab'$.

$$(6)\quad \frac{x^2-px+q}{x+a} = x-(a+p) + \frac{k}{x} - \frac{ak}{x^2} + \frac{a^2k}{x^3} - \&c.$$

where $k = a^2 + ap + q$.

INVOLUTION AND EVOLUTION.

82. By *Involution* is meant the raising of a quantity to any proposed power; and by *Evolution*, the extraction of roots, or the determining of the quantity which raised to a proposed power will produce a given quantity. (Arts. 19 and 21.)

83. If a quantity to be involved be negative, the signs of the even powers will be positive, and the signs of the odd powers negative.

For $(-a)(-a)=a^2$, $(-a)(-a)(-a)=-a^3$, &c.

84. Hence, if a root to be extracted be indicated by an odd number, the sign of the root will be the same as the sign of the proposed quantity; also if the root be indicated by an even number and the proposed quantity be positive, the root may be either positive or negative.

85. If the radical sign with an even index be prefixed to a negative quantity, the root indicated cannot be extracted; because no quantity raised to an even power can produce a negative result.

86. A simple quantity, or monomial, is raised to any power by multiplying the index of every factor in the quantity by the exponent of the power, and prefixing the proper sign.

Thus $(ab)^m=(ab)\,(ab)\,(ab)\,\ldots\ldots$ to m factors

$$=\{a.a.a\ldots\ldots(\text{to } m \text{ factors})\}.\{b.b.b\ldots\ (\text{to } m \text{ factors})\}$$

$$=a^m b^m$$

Also $(a^m)^n = a^{mn}$;

for $(a^m)^n = a^m a^m a^m \ldots\ldots (n \text{ times})$

$= a^{m+m+m+\ldots (n \text{ times})}$ (Art. 45),

$= a^{mn}$.

Hence $(a^m b^n)^p = (a^m)^p \cdot (b^n)^p = a^{mp} b^{np}$.

87. In Art. 45 it is shewn that $a^m a^n = a^{m+n}$, m and n being positive integers.

Let m and n be fractions, $m = \frac{p}{q}$ and $n = \frac{r}{s}$, p, q, r, and s being positive integers.

Then also $a^{\frac{p}{q}} a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$,

for $a^{\frac{p}{q}} = a^{\frac{ps}{qs}} = (a^{\frac{1}{qs}})^{ps}$,

and $a^{\frac{r}{s}} = a^{\frac{qr}{qs}} = (a^{\frac{1}{qs}})^{qr}$,

$\therefore\ a^{\frac{p}{q}} a^{\frac{r}{s}} = (a^{\frac{1}{qs}})^{ps+qr}$

$= a^{\frac{ps+qr}{qs}}$, or $a^{\frac{p}{q}+\frac{r}{s}}$.

88. Any root of a simple quantity is found by dividing the index of each factor by the exponent of the root required, and prefixing the proper sign.

Thus, $(a^m b^n)^{\frac{1}{p}} = a^{\frac{m}{p}} b^{\frac{n}{p}}$; because each of these quantities raised to the p^{th} power is $a^m b^n$.

Also $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$;

for $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}}$, raised to the power of s, $=\left(a^{\frac{p}{q}}\right)^{r}=a^{\frac{pr}{q}}$,

and $a^{\frac{pr}{qs}}$, raised to the power of s, $=a^{\frac{pr}{q}}$;

$$\therefore\ (a^{\frac{p}{q}})^{\frac{r}{s}}=a^{\frac{pr}{qs}}.$$

Hence also

$$\left(a^{\frac{m}{n}}b^{\frac{p}{q}}\right)^{\frac{r}{s}}=\left(a^{\frac{m}{n}}\right)^{\frac{r}{s}}.\left(b^{\frac{p}{q}}\right)^{\frac{r}{s}}=a^{\frac{mr}{ns}}b^{\frac{pr}{qs}}.$$

89. Any power or root of a fraction is found by taking that power or root of both the numerator and denominator of the fraction, (Art. 76).

The following examples will illustrate the above articles.

(1) $\{(a^m)^n\}^p=(a^{mn})^p=a^{mnp}$.

(2) $(a^{-1})^{-2}=a^2$.

(3) $[\{(a^m)^{-n}\}^{-p}]^{-q}=\{(a^{-mn})^{-p}\}^{-q}=(a^{mnp})^{-q}=a^{-mnpq}$ or $\frac{1}{a^{mnpq}}$, (Art. 51).

(4) $\{(-a)^{-2n}\}^{-2n}=(a^{-2n})^{-2n}=a^{4n^2}$, (Art. 83).

(5) $\{(-a)^{-(2n+1)}\}^{-(2n+1)}=(-a^{-(2n+1)})^{-(2n+1)}=-a^{(2n+1)^2}$ $=-a^{4n^2+4n+1}$.

(6) $\sqrt{\sqrt[3]{\sqrt[4]{a}}}=\{(a^{\frac{1}{4}})^{\frac{1}{3}}\}^{\frac{1}{2}}=a^{\frac{1}{24}}$.

(7) $(3ab^2)^3=3^3a^3b^6=27a^3b^6$.

(8) $(a^m\times 3b^n\times 4c^{-p}\times 5d^{-q})^2=3^2\times 4^2\times 5^2a^{2m}b^{2n}c^{-2p}d^{-2q}$

$$=60^2\frac{a^{2m}b^{2n}}{c^{2p}d^{2q}}=3600\frac{a^{2m}b^{2n}}{c^{2p}d^{2q}}.$$

(9) $\left\{\left(-\frac{a}{b}\right)^3\right\}^{-4}=\left(-\frac{a^3}{b^3}\right)^{-4}=\left(-\frac{b^3}{a^3}\right)^4=\frac{b^{12}}{a^{12}}$.

(10) $\{(-ab^2)^3\}^4=(-a^3b^6)^4=a^{12}b^{24}$.

(11) $(a^3b^{-3}c^6d^{-6})^{\frac{1}{3}} = \dfrac{ac^2}{bd^2}$.

(12) $(a^{\frac{3}{4}}b^{\frac{1}{3}})^{\frac{2}{3}} = a^{\frac{1}{2}}b^{\frac{2}{9}}$, or $(a^9b^4)^{\frac{1}{18}}$.

(13) $3^{\frac{1}{2}} \times 4^{\frac{1}{3}} = 3^{\frac{3}{6}} \times 4^{\frac{2}{6}} = (27)^{\frac{1}{6}} \times (16)^{\frac{1}{6}} = (432)^{\frac{1}{6}}$.

(14) $(a^{\frac{2}{3}} + 3a^{\frac{1}{3}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}}) \times (a^{\frac{1}{3}} - 3b^{\frac{1}{3}}) = a - 27b$.

(15) $x^{\frac{3}{2}} - y^{\frac{3}{2}} \div x^{\frac{1}{2}} - y^{\frac{1}{2}} = x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

90. *To extract the square root of a compound quantity.*

By observing in what manner $a + b$ can be derived from its square $a^2 + 2ab + b^2$, we may obtain a general rule for the extraction of the square root.

$$\begin{array}{r l}
 & a^2 + 2ab + b^2 \;(a + b \\
 & a^2 \\
\hline
2a + b) & 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
 & \cdot \quad \cdot
\end{array}$$

Having arranged the terms according to the dimensions of a, we observe that the square root of the first term (a^2) is a, the first term of the root; subtracting its square from the whole quantity and dividing $2ab$, the first term of the remainder, by twice the first term of the root, we obtain b, the second term of the root; adding b to $2a$, proceeding with $2a + b$ as a divisor and b as a quotient, and subtracting their product from the last remainder, we have no further remainder: and a^2 in the first place, and then $(2a + b)\,b$ or $2ab + b^2$, which together are equal to $(a + b)^2$, having been subtracted from the given square, we know that the root thus found is accurate.

Whenever there is a remainder, the process must be repeated, the terms of the root already found being considered as a single term: and if there be perpetually a remainder,

the given quantity is not a complete square, but there will be found, for an approximation to the root, an interminable series similar to the incomplete quotient in division.

The nature of the process being the same whatever be the example proposed, the explanation of the above operation will serve for a general rule.

(2) Let it be required to extract the square root of

$$a^2+2ab+b^2+2ac+2bc+c^2.$$

$$\begin{array}{rl}
 & a^2+2ab+b^2+2ac+2bc+c^2 \;(a+b+c \\
 & a^2 \\
\hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
2a+2b+c) & 2ac+2bc+c^2 \\
 & 2ac+2bc+c^2 \\
\hline
 & \cdot \quad \cdot \quad \cdot
\end{array}$$

In this example we have subtracted from the given square, by two operations, the square of $a+b$, and by the third operation, $(2a+2b+c)c$, which added to $(a+b)^2$ makes the complete square of $a+b+c$.

(3) Extract the square root of $a^2-ax+\frac{x^2}{4}$.

$$\begin{array}{rl}
 & a^2-ax+\frac{x^2}{4} \;(a-\frac{x}{2} \\
 & a^2 \\
\hline
2a-\frac{x}{2}) & -ax+\frac{x^2}{4} \\
 & -ax+\frac{x^2}{4} \\
\hline
 & \cdot \quad \cdot
\end{array}$$

If we had reversed the order of the terms, we should have obtained $\frac{x}{2} - a$, or $-\left(a - \frac{x}{2}\right)$, for the root; the result in every instance, in passing from the square to the root, admitting of either sign, $+$ or $-$, (Art. 66).

(4) Extract the square root of $\frac{a^2}{b^2} + \frac{b^2}{a^2} - 2$.

$$\begin{array}{r|l}
 & \frac{a^2}{b^2} - 2 + \frac{b^2}{a^2} \left(\frac{a}{b} - \frac{b}{a} \right. \\
 & \frac{a^2}{b^2} \\ \hline
\left. \frac{2a}{b} - \frac{b}{a} \right) & - 2 + \frac{b^2}{a^2} \\
 & - 2 + \frac{b^2}{a^2} \\ \hline
 & \cdot \quad \cdot
\end{array}$$

In arranging the terms according to the powers of a, $\frac{b^2}{a^2}$ must be placed after -2, because it involves a negative power of a: this will perhaps be more easily seen if the expression be put under the form $a^2b^{-2} - 2a^0b^0 + a^{-2}b^2$. The contrary arrangement would equally answer the purpose; but the neglect of such arrangement might cause the process to be interminable.

(5) Extract the square root of $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2$.

Arranging it according to the powers of a, we have

$$\begin{array}{r|l}
 & 9a^2 - 6(b - 5c - d)a + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2 \left(3a - b + 5c + d \right. \\
 & 9a^2 \\ \hline
6a - (b - 5c - d)) & -6(b - 5c - d)a + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2 \\
 & -6(b - 5c - d)a + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2 \\ \hline
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

(6) Extract the square root of $\dfrac{a^2x^2 + 2ab^2x^3 + b^4x^4}{a^{2m} + 2a^mx^n + x^{2n}}$.

Extracting the root of the numerator and denominator separately, we have

$$\begin{array}{rl}
 & a^2x^2 + 2ab^2x^3 + b^4x^4 \;(ax + b^2x^2 \\
 & a^2x^2 \\
\hline
2ax + b^2x^2) & 2ab^2x^3 + b^4x^4 \\
 & 2ab^2x^3 + b^4x^4 \\
\hline
 & \cdot \qquad \cdot
\end{array}$$

and

$$\begin{array}{rl}
 & a^{2m} + 2a^mx^n + x^{2n} \;(a^m + x^n \\
 & a^{2m} \\
\hline
2a^m + x^n) & 2a^mx^n + x^{2n} \\
 & 2a^mx^n + x^{2n} \\
\hline
 & \cdot \qquad \cdot
\end{array}$$

Whence the required root is $\dfrac{ax + b^2x^2}{a^m + x^n}$.

(7) Let it be required to extract the square root of $1 + x$.

$$\begin{array}{rl}
 & 1 + x \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \text{\&c.}\right. \\
 & 1 \\
\hline
2 + \frac{x}{2}) & x \\
 & x + \frac{x^2}{4} \\
\hline
2 + x - \frac{x^2}{8}) & -\frac{x^2}{4} \\
 & -\frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
\hline
2 + x - \frac{x^2}{4} + \frac{x^3}{16}) & \frac{x^3}{8} - \frac{x^4}{64} \\
 & \frac{x^3}{8} + \frac{x^4}{16} - \frac{x^5}{64} + \frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64} + \frac{x^5}{64} - \frac{x^6}{256}
\end{array}$$

If we make x the first term, we shall obtain instead of the above, the following series,

$$x^{\frac{1}{2}} + \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{8x^{\frac{3}{2}}} + \&c.$$

But the involution and evolution of binomials is effected most easily by the *Binomial Theorem.*

EXAMPLES FOR PRACTICE.

(1) Shew that $\sqrt{\left(\frac{a^2-4a+4}{a^{2m}-4a^{m+n}+4a^{2n}}\right)} = \frac{a-2}{a^m-2a^n}$.

(2) $\sqrt{\left(x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}\right)} = x^2-x+\frac{1}{4}$.

(3) $\sqrt{\left\{x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1\right\}} = x+1-\frac{1}{x}$.

(4) $\sqrt{\left(\frac{4x^2}{49y^2}-\frac{20x}{7y}-\frac{15y}{2x}+\frac{9y^2}{16x^2}+25\frac{3}{7}\right)} = \frac{2x}{7y}-5+\frac{3y}{4x}$.

(5) $\sqrt{\left(25a^{2m-4}c^2x^{2n+2}+10a^{2m-2}cx^{2n+1}+a^{2m}x^{2n}\right.}$
$\left.-30a^{m-1}cx^n-6a^{m+1}x^{n-1}+\frac{9a^2}{x^2}\right) = 5a^{m-2}cx^{n+1}+a^mx^n-\frac{3a}{x}$.

91. *To extract the square root in numbers.*

Since the square of $10 = 100$, of $100 = 10000$, of $1000 = 1000000$, &c., the roots of all square numbers containing one or two, three or four, five or six, ..., $2n-1$ or $2n$ figures, consist of one, two, three, ..., n figures respectively: if, therefore, a point be placed over every alternate figure, beginning with the units, the number of periods thus marked off will shew the number of figures in the root: thus, the square root of $4\dot{0}9\dot{6}$ consists of two figures, the square root of $\dot{6}10\dot{0}9$ of three figures, &c.

(1) Let the square root of 956484 be required.

$$\begin{array}{rl}
 & \dot{9}5\dot{6}4\dot{8}4\ (\overset{a}{900} + \overset{b}{70} + \overset{c}{8} \\
 & \underline{810000} = a^2 \\
2a+b=1800+70=1870) & 146484 \\
 & \underline{130900} = (2a+b)\,b \\
2a+2b+c=1940+8=1948) & 15584 \\
 & \underline{15584} = (2a+2b+c)\,c \\
 & \cdots\cdots
\end{array}$$

Here the points indicate that the root consists of three figures, which we suppose to be a, b, and c, where a is the value of the figure in the place of hundreds, b of that in the place of tens, and c of that in the place of units. Then 81 being the largest square number in the first period (95), 900 (a) is the nearest square root of 950000, which does not exceed the true root; subtract the square of this (a^2) from the given number, and divide the remainder by 1800 ($2a$) in order to obtain the value of b, which appears to be 70; whence the next quantity to be subtracted is 1870×70, $(2a+b)\,b$, or 130900: and proceeding with $2(a+b)$ as with $2a$, we find c the remaining figure of the root.

Omitting the ciphers for the sake of expedition, we obtain from the foregoing process the following rule.

$$\begin{array}{rl}
 & \dot{9}5\dot{6}4\dot{8}4\ (978 \\
 & \underline{81} \\
187) & 1464 \\
 & \underline{1309} \\
1948) & 15584 \\
 & \underline{15584} \\
 & \cdots\cdots
\end{array}$$

First ascertain the number of figures in the root by placing a full-point over every alternate figure, beginning with the units: for the first figure in the root, find the greatest number whose square is contained in the first period: subtract its square from the first period, and to the remainder bring down the next period: with twice the part of the

root already found, as a divisor, find by trial the next figure of the root, and place it both in the divisor and in the root; then multiply the whole divisor, thus completed, by the part of the root last obtained, subtract the product thus found, and to the remainder, if any, bring down another period: proceed with twice the part of the root already found as before, and in this manner repeat the process as often as required.

(2) Extract the square root of 7513081.

```
          7513081 (2741
          4
          ---
    47)   351
          329
          ----
   544)   2230
          2176
          -----
  5481)   5481
          5481
          ------
          ......
```

92. In extracting the square root of a decimal, the pointing must be made in the contrary direction, beginning with the hundredths, care being taken to have an even number of decimal places.

(3) Extract the square root of .582169.

```
          .582169 (.763
           49
           ---
   146)    921
           876
           ----
  1523)    4569
           4569
           ------
           ......
```

(4) Extract the square root of .0081.

$$\begin{array}{l} .00\dot{8}1\ (.09 \\ \underline{81} \\ \ldots \end{array}$$

(5) Let the square root of 64.853 be required.

$$\begin{array}{rl} & \dot{6}4.\dot{8}5\dot{3}0\ (8.053\ \&c. \\ & \underline{64} \\ 1605) & 8530 \\ & \underline{8025} \\ 16103) & 50500 \\ & \underline{48309} \\ & 2191 \end{array}$$

For every pair of ciphers which we suppose added to the decimal, another figure is obtained in the root: and by continuing the process it is obvious that we shall continually approximate to the value of the root.

In the same manner, by annexing pairs of ciphers, with the decimal point prefixed, to any proposed integer which is not a complete square, we may continue the operation indefinitely.

(6) Extract the square root of 5 as far as three places of decimals.

$$\begin{array}{rl} & \dot{5}.\dot{0}0\ \ldots\ (2.236\ldots \\ & \underline{4} \\ 42) & 100 \\ & \underline{84} \\ 443) & 1600 \\ & \underline{1329} \\ 4466) & 27100 \\ & \underline{26796} \\ & 304 \end{array}$$

EXAMPLES FOR PRACTICE.

(1) Shew that $\sqrt{106929}=327$.

(2) $\sqrt{824464}=908$.

(3) $\sqrt{81144064}=9008$.

(4) $\sqrt{41605800625}=203975$.

(5) $\sqrt{.0081}=.09$.

(6) $\sqrt{.000625}=.025$.

(7) $\sqrt{.004}=.06324 \ldots$

(8) $\sqrt{.00038}=.01949\ldots\ldots$

(9) $\sqrt{\frac{4096}{582169}}=\frac{64}{763}$.

(10) $\sqrt{\frac{5}{12}}=\sqrt{.41666\ldots}=.64549\ldots$

(11) $\sqrt{7\frac{13}{36}}=\sqrt{7.36111\ldots}=2.713136\ldots$

(12) $\sqrt{.444\ldots}=.666\ldots$: or, $\sqrt{.444\ldots}=\sqrt{\frac{4}{9}}=\frac{2}{3}=.666\ldots$

93. *To extract the cube root of a compound algebraical quantity.*

A method of extracting the cube root may be derived from the cube of $a+b$ in the same manner as the method of extracting the square root was from the square of $a+b$; thus, since we have $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, the inverse process will stand as follows:

$$\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3\ (a+b \\
 & a^3 \\
\hline
3a^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
 & \cdot \quad \cdot \quad \cdot
\end{array}$$

The terms being arranged according to the powers of some letter, we find the cube root of the first term for the first term of the root, and subtract its cube: then take three times its square for an imperfect divisor, and divide the first term of the remainder by it, in order to obtain the second term of the root: we then take away the remaining part of the cube of the terms in the root; namely, three times the square of the first term multiplied into the second, three times the first term into the square of the second, and the cube of the second term: and whenever there is a remainder, proceed as before, considering the terms already in the root as constituting one term.

(2) Extract the cube root of

$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1.$$

$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1\ (x^2-2x+1$$
$$x^6$$
$$3x^4)\ -6x^5+15x^4-20x^3$$
$$-6x^5+12x^4-\ 8x^3=3x^4(-2x)+3x^2(-2x)^2+(-2x)^3$$
$$\left.\begin{matrix}3\,(x^2-2x)^2=\\3x^4-12x^3+12x^2\end{matrix}\right)\ 3x^4-12x^3+15x^2-6x+1$$
$$3x^4-12x^3+15x^2-6x+1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(3) Extract the cube root of $\left(x^3+\frac{1}{x^3}\right)+3\left(x+\frac{1}{x}\right)$.

$$x^3+3x+\frac{3}{x}+\frac{1}{x^3}\ \left(x+\frac{1}{x}\right.$$
$$x^3$$
$$3x^2\Big)\ 3x+\frac{3}{x}+\frac{1}{x^3}$$
$$3x+\frac{3}{x}+\frac{1}{x^3}$$
$$\cdot \quad \cdot \quad \cdot$$

EXAMPLES FOR PRACTICE.

(1) Shew that $\sqrt[3]{(8x^6+48ax^5+60a^2x^4-80a^3x^3-90a^4x^2+108a^5x-27a^6)}=2x^2+4ax-3a^2$.

(2) $\sqrt[3]{\left(\frac{x^3}{y^6}-\frac{3x}{y^2}-\frac{y^6}{x^3}+\frac{3y^2}{x}\right)}=\frac{x}{y^2}-\frac{y^2}{x}$.

(3) $\sqrt[3]{(a^{3m}-6a^{2m+1}x^n+12a^{m+2}x^{2n}-8a^3x^{3n})}=a^m-2ax^n$.

94. *To extract the cube root in numbers.*

Since the cube of $10=1000$, of $100=1000000$, &c., if a cube number have $3n-2$, $3n-1$, or $3n$ figures, its root must have n figures: if, therefore, a point be made over every third figure of any cube number, beginning with the units, the number of points will shew the number of figures in its root.

(1) Let the cube root of 40353607 be required.

$$
\begin{array}{r|l}
 & \dot{4}035\dot{3}60\dot{7}\;(\overset{a}{300}+\overset{b}{40}+\overset{c}{3} \\
 & \underline{27000000}\;(S_1)=a^3 \\
3a^2=270000 & 13353607\;R_1 \\
(3a+b)\,b=\underline{\;\;37600} & \\
3a^2+(3a+b)\,b=307600 & 12304000\;(S_2)=3a^2b+3ab^2+b^3 \\
b^2=\;\;\;1600 & \overline{\;1049607}\;R_2 \\
3a_1^2=346800 & \\
(3a_1+c)\,c=\underline{\;\;\;3069} & \\
3a_1^2+(3a_1+c)c=349869 & \underline{\;1049607}\;(S_3)=3a_1^2c+3a_1c^2+c^3 \\
 & \;\;\cdots\cdots
\end{array}
$$

Here the points indicate that the root consists of three figures, whose values we suppose to be a, b, and c, respectively: first subtracting a^3, which is the greatest cube contained in 40000000; we divide the remainder (R_1) by $3a^2$,

in order to obtain b; and to $3a^2$ add $(3a+b)\,b$, multiply their sum by b, and subtract the product (S_2), which is the remaining part of the cube of $a+b$: then proceeding as before with $3\,(a+b)^2$, which is the sum of b^2 and the two lines standing over it, for our second divisor, representing it by $3a_1^2$, we obtain c, and (S_3), which completes the cube of

$$a_1+c \text{ or of } a+b+c:$$

and since the sum

$$(S_1+S_2+S_3)=(a+b)^3+3\,(a+b)^2\,c+3\,(a+b)\,c^2+c^3$$
$$\text{or } \{(a+b)+c\}^3,$$

and there is no remainder, the root thus found is accurate.

The above operation may be conducted also under the following abbreviated form, according to the rule subjoined.

		$4\dot{0}35\dot{3}60\dot{7}$ (343
		27
94	27	13353
	376	
	3076	12304
	16	
1023	3468	1049607
	3069	
	349869	1049607
		......

Having ascertained the number of figures in the root by placing a point over every third figure of the given number, beginning with the units, the first figure in the root is the greatest number whose cube is contained in the first period on the left hand; find this, subtract its cube from the first

period, and to the remainder bring down the next period; the number thus formed is commonly called the resolvend: some way to the left of the resolvend place three times the root already found (9), and multiply it by the root for a partial divisor; then divide the resolvend, omitting the last two figures, by this divisor (27), and make the quotient, taken in defect if necessary, the next figure of the root; annex it (the quotient) also to the triple of the root, and multiply the number thus formed (94) by it; place the product under the divisor, moving it two places to the right, because there are two ciphers more omitted in the divisor than in it, add the two together, and multiply their sum by the quotient (4); subtract the product from the resolvend above, and to the remainder bring down the next period; for another divisor take the sum of the two lines which stand immediately under the preceding divisor and the square of the figure of the root last found, proceed with this as before, and repeat the process as often as required.

95. When the cube root of a decimal is required, periods of three places must be marked off to the right of the units, as many zeros being annexed as may be necessary to make the number of decimal places 3, or a multiple of 3; because there are three times as many decimal places in the cube as in the root.

96. Upon the same principle, the extraction of the cube root either of an integer or decimal, which is not a complete cube, may be continued indefinitely, by annexing periods of zeros continually, considering them, and all the places in the root corresponding to them, as decimal places: we may thus approximate as near as we please to the required root.

(2) Let the cube root of 102.87 be required.

		$10\dot{2}.87\dot{0}$ (4.685....
		64
126	48	38870
	756	
	5556	33336
	36	
1388	6348	5534000
	11104	
	645904	5167232
	64	
14045	657072	366768000
	70225	
	65777425	328887125
		37880875

EXAMPLES FOR PRACTICE.

(1) Shew that $\sqrt[3]{68921}=41$.

(2) $\sqrt[3]{17173512}=258$.

(3) $\sqrt[3]{2197000}=130$.

(4) $\sqrt[3]{.001}=.1$, and $\sqrt[3]{.000001}=.01$.

(5) $\sqrt[3]{10}=2.154\ldots\ldots\sqrt[3]{.01}=.2154\ldots\ldots$, and $\sqrt[3]{.00001}=.02154\ldots\ldots$

(6) $\sqrt[3]{100}=4.641\ldots\ldots$, $\sqrt[3]{.1}=.4601\ldots\ldots$, and $\sqrt[3]{.0001}=.04641\ldots\ldots$

(7) $\sqrt[3]{48230.605376}=38.76$.

(8) $\sqrt[3]{4}=1.5874\ldots\ldots$

(9) $\sqrt[3]{\dfrac{343}{512}}=\dfrac{7}{8}$.

(10) $\sqrt[3]{\frac{5}{6}} = \sqrt[3]{.8333 \ldots\ldots} = .94103 \ldots\ldots$

(11) $\sqrt[3]{\frac{82}{216}} = \frac{\sqrt[3]{82}}{6} = \frac{4.34448\ldots\ldots}{6} = .72408\ldots\ldots$

(12) $\sqrt[3]{52034\frac{10}{27}} = \frac{\sqrt[3]{1404928}}{3} = \frac{112}{3} = 37\frac{1}{3}$.

Or thus, $\sqrt[3]{52034\frac{10}{27}} = \sqrt[3]{52034.\dot{3}7\dot{0}} = 37.\dot{3} = 37\frac{1}{3}$.

97. Similar methods may be applied to the extraction of the roots of higher powers, but they are seldom employed in practice.

The fourth root may be found by taking the square root of the square root, the sixth root by taking the cube root of the square root, and so on for all composite indices: for $a^{\frac{1}{4}} = (a^{\frac{1}{2}})^{\frac{1}{2}}$, $a^{\frac{1}{6}} = (a^{\frac{1}{2}})^{\frac{1}{3}}$, &c.

The roots of the higher powers of numbers are extracted by means of logarithms, even in the case of the cube root when the cube contains several periods, more rapidly than by any other method.

SURDS.

98. Quantities affected by a radical sign, or fractional index, denoting a root which cannot be extracted in a finite number of terms, are called *Surds* or *Irrational quantities*.

Thus $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{a}$, $\sqrt{(1+x)}$, $\sqrt{(a+x)^3}$ and $\left(\frac{a \sim x}{a+x}\right)^{\frac{2}{3}}$ are surds.

99. Since $a = a^1 = a^{\frac{2}{2}} = a^{\frac{3}{3}} = a^{\frac{m}{m}}$, and $a^{\frac{2}{2}} = \sqrt{a^2}$,

$$a^{\frac{3}{3}} = \sqrt[3]{a^3},\ a^{\frac{m}{m}} = \sqrt[m]{a^m},$$

a rational quantity may be represented as a surd, by raising

it to the power whose root the surd expresses, and affixing the radical sign, or denoting the root by the denominator of a fractional index.

In the same manner the form of any surd may be altered; thus

$$(a+x)^{\frac{1}{2}}=(a+x)^{\frac{2}{4}}=(a+x)^{\frac{3}{6}}=\frac{1}{(a+x)^{-\frac{1}{2}}}=\frac{1}{(a+x)^{-\frac{2}{4}}}$$

$$=\left\{\frac{1}{(a+x)^{-2}}\right\}^{\frac{1}{4}}=\left\{\frac{1}{(a+x)^{-3}}\right\}^{\frac{1}{6}}=\&c.$$

100. Since $(a^m b)^{\frac{1}{m}} = a^{\frac{m}{m}} b^{\frac{1}{m}} = ab^{\frac{1}{m}}$, if a quantity under a radical sign or fractional index contain a factor which is a perfect power corresponding to the root indicated, its root may be extracted and made a rational factor of the remaining surd.

EXAMPLES.

(1) $\sqrt{(27x^3)}=\sqrt{(3^2\times 3x^2x)}=3x\sqrt{(3x)}$.

(2) $(108a^3x)^{\frac{1}{3}}=(27\times 4a^3x)^{\frac{1}{3}}=3a(4x)^{\frac{1}{3}}$.

(3) $\sqrt{(72x+108y)}=\sqrt{(36\times 2x+36\times 3y)}=$ $6\sqrt{(2x+3y)}$.

(4) $\sqrt{\dfrac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}}=\dfrac{a-x}{a+x}\sqrt{x}$.

(5) $\left(\dfrac{a^{2mn}b^{2n+1}}{c^{4n-3}d^{4mn}}\right)^{\frac{1}{2n}}=\dfrac{a^m b}{c^2 d^{2m}}(bc^3)^{\frac{1}{2n}}$.

(6) $\sqrt{\left(\dfrac{a^3b^2}{c\,d^2}-\dfrac{a^2b^3}{c^2d}\right)}=\sqrt{\dfrac{a^3b^2c-a^2b^3d}{c^2d^2}}$
$=\dfrac{ab}{cd}\sqrt{(ac-bd)}$.

(7) $\sqrt{\dfrac{a^2x^2}{a-x}}=\dfrac{ax}{\sqrt{(a-x)}}$ or $\dfrac{ax}{a-x}\sqrt{(a-x)}$.

101. Conversely any rational factor or coefficient of a surd may be introduced under the radical sign, or fractional index, when raised to the power whose root is indicated by the surd.

(1) $\sqrt{(5\sqrt[3]{2})} = \{(5^3 \times 2)^{\frac{1}{3}}\}^{\frac{1}{2}} = (250)^{\frac{1}{6}}$.

(2) $\dfrac{a+x}{a-x}\sqrt{\dfrac{a-x}{a+x}} = \sqrt{\dfrac{(a+x)^2(a-x)}{(a-x)^2(a+x)}} = \sqrt{\dfrac{a+x}{a-x}}$.

ADDITION AND SUBTRACTION OF SURDS.

102. The sum or difference of surds which have the same irrational part, is found by affixing the sum or difference of their rational factors, or coefficients, to that irrational part. Surds which have not the same irrational part are connected together by means of the proper signs.

EXAMPLES.

(1) $a\sqrt{x} - b\sqrt{x} + c\sqrt{x} = (a-b+c)\sqrt{x}$.

(2) $\sqrt{(45c^3)} - \sqrt{(80c^3)} + \sqrt{(5a^2c)} = 3c\sqrt{(5c)} - 4c\sqrt{(5c)} + a\sqrt{(5c)} = (a-c)\sqrt{5c}$.

(3) $(16a^3b)^{\frac{1}{3}} + (a^2b)^{\frac{1}{2}} - (54a^3b)^{\frac{1}{3}} = ab^{\frac{1}{2}} - a(2b)^{\frac{1}{3}}$.

(4) $a^3bc(a^{-9}bc)^{\frac{1}{5}} - b^2c(a^6b^{-4}c)^{\frac{1}{5}} + a^2b^4c^2(243a^{-4}b^{-14}c^{-4})^{\frac{1}{5}} = 3abc(abc)^{\frac{1}{5}}$.

(5) $(54a^{m+6}b^3)^{\frac{1}{3}} - (16a^{m-3}b^6)^{\frac{1}{3}} + (2a^{4m+9})^{\frac{1}{3}} + (2c^3a^m)^{\frac{1}{3}} = \left(3a^2b - \dfrac{2b^2}{a} + a^{m+3} + c\right)(2a^m)^{\frac{1}{3}}$.

MULTIPLICATION OF SURDS.

103. Since $a^{\frac{m}{p}} b^{\frac{n}{p}} = (a^m b^n)^{\frac{1}{p}}$, if the indices of two quantities have a common denominator, their product is found by raising them to the powers expressed by their respective numerators, and extracting the root indicated by the common denominator.

If the indices have not a common denominator, transform them so as to have a common denominator, and proceed as before:

$$\text{thus, } a^{\frac{1}{m}} b^{\frac{1}{n}} = a^{\frac{n}{mn}} b^{\frac{m}{mn}} = (a^n b^m)^{\frac{1}{mn}}.$$

If the surds have the same rational quantity under the radical signs, or fractional indices, their product is found by making the sum of the indices the index of that quantity, (Art. 45).

If the surds have rational factors, their product must be prefixed; thus, $(a\sqrt{x})(b\sqrt{y}) = ab\sqrt{(xy)}$.

EXAMPLES.

(1) $\sqrt{3} \times \sqrt{8} = \sqrt{24} = \sqrt{(4 \times 6)} = 2\sqrt{6}$.

(2) To find the product of $3\sqrt{2} - \sqrt{3}$ and $3\sqrt{3} + 4\sqrt{2}$.

$$\begin{array}{l} 3\sqrt{2} - \sqrt{3} \\ 3\sqrt{3} + 4\sqrt{2} \\ \hline 9\sqrt{6} - \quad 3 \times 3 \\ \qquad\quad 12 \times 2 - 4\sqrt{6} \\ \hline 9\sqrt{6} + 15 - 4\sqrt{6} = 5\sqrt{6} + 15 \end{array}$$

(3) $5\sqrt{3} \times 7\sqrt{\frac{8}{3}} \times \sqrt{2} = 140$.

(4) $(5\sqrt{14} + 3\sqrt{5})(7\sqrt{14} - 2\sqrt{5}) = 460 + 11\sqrt{70}$.

(5) $5\sqrt{2}\times 3\sqrt{(4+6\sqrt{2})} = 30\sqrt{(2+3\sqrt{2})}$.

(6) $(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}})^2 = a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}+2(ab)^{\frac{1}{4}}+2(ac)^{\frac{1}{4}}$
$+2(bc)^{\frac{1}{4}}$.

(7) $(a+b^{\frac{1}{n}})^{\frac{1}{m}}(c+d^{\frac{1}{p}})^{\frac{1}{m}} = \{ac+cb^{\frac{1}{n}}+ad^{\frac{1}{p}}+(b^p d^n)^{\frac{1}{np}}\}^{\frac{1}{m}}$.

(8) $\{x^{\frac{1}{2}}+(xy)^{\frac{1}{4}}+y^{\frac{1}{2}}\}\{x^{\frac{1}{2}}-(xy)^{\frac{1}{4}}+y^{\frac{1}{2}}\} =$

$(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2-(xy)^{\frac{1}{2}} = x+(xy)^{\frac{1}{2}}+y$.

(9) $(a+\sqrt{b})^{\frac{1}{n}}(a-\sqrt{b})^{\frac{1}{n}}=(a^2-b)^{\frac{1}{n}}$.

(10) $(9+2\sqrt{10})(9-2\sqrt{10})=41$.

(11) $\left\{\left(\frac{ax^2}{b^3}\right)^{\frac{1}{2}}+\left(\frac{c}{d}\right)^{\frac{1}{2}}\right\}\left\{\frac{x}{b}\left(\frac{a}{b}\right)^{\frac{1}{2}}-\left(\frac{c}{d}\right)^{\frac{1}{2}}\right\}=\frac{ax^2}{b^3}-\frac{c}{d}$.

(12) $\{\sqrt{a}-\sqrt{(b-\sqrt{c})}\}\{\sqrt{a}+\sqrt{(b-\sqrt{c})}\}=a-b+\sqrt{c}$.

(13) $(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}+\sqrt{c}-\sqrt{b})(\sqrt{b}+\sqrt{c}-\sqrt{a})$
$=2(ab+ac+bc)-a^2-b^2-c^2$.

DIVISION OF SURDS.

104. Since $\dfrac{a^{\frac{m}{p}}}{b^{\frac{n}{p}}}=\dfrac{(a^m)^{\frac{1}{p}}}{(b^n)^{\frac{1}{p}}}=\left(\dfrac{a^m}{b^n}\right)^{\frac{1}{p}}$, (Art. 76), if the indices of two quantities have a common denominator, the quotient of one divided by the other is obtained by raising them, respectively, to the powers expressed by the numerators of their indices, and extracting that root of the quotient which is indicated by the common denominator.

If the indices have not a common denominator, transform them so as to have a common denominator, and proceed as before: thus,

$$\frac{a^{\frac{1}{m}}}{b^{\frac{1}{n}}} = \frac{a^{\frac{n}{mn}}}{b^{\frac{m}{mn}}} = \frac{(a^n)^{\frac{1}{mn}}}{(b^m)^{\frac{1}{mn}}} = \left(\frac{a^n}{b^m}\right)^{\frac{1}{mn}}.$$

If the surds have the same rational quantity under the radical signs, their quotient is obtained by making the difference of their indices the index of that quantity; thus,

$$\frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}} \text{ or } \frac{a^{\frac{m}{mn}}}{a^{\frac{n}{mn}}} = a^{\frac{m-n}{mn}};$$

because these quantities, raised to the power mn, produce equal results, viz. $\frac{a^m}{a^n}$ and a^{m-n} (Art. 51).

If the surds have rational factors, their quotient must be prefixed; thus,

$$\frac{a\sqrt{x}}{b\sqrt{y}} = \frac{a}{b}\sqrt{\frac{x}{y}}.$$

EXAMPLES.

(1) $(\sqrt{72}+\sqrt{32}-4)\div\sqrt{8}=5-\sqrt{2}$.

(2) To find the quotient of $5\sqrt{6}+15$ by $3\sqrt{2}-\sqrt{3}$.

To find this quotient by division, it would be necessary to restore the original form of the product in Ex. 2, Art. 103; but as the original form of the product of two quantities involving the roots of the same numbers is not easily discovered, recourse is commonly had to the following process of rationalizing binomial quadratic surds, which is furnished by the theorem given in Ex. 4, p. 16:—

$$\frac{5\sqrt{6}+15}{3\sqrt{2}-\sqrt{3}} = \frac{5\sqrt{6}+15}{3\sqrt{2}-\sqrt{3}} \times \frac{3\sqrt{2}+\sqrt{3}}{3\sqrt{2}+\sqrt{3}} = \frac{45\sqrt{3}+60\sqrt{2}}{15} = 3\sqrt{3}+4\sqrt{2}.$$

(3) To find the quotient of $1-\sqrt{3}-\sqrt{2}$ by $\sqrt{3}-\sqrt{2}-1$.

Treating $\sqrt{3}-\sqrt{2}$ as one term,

$$\frac{1-\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}-1}=\frac{1-(\sqrt{3}+\sqrt{2})}{\sqrt{3}-\sqrt{2}-1}\times\frac{\sqrt{3}-\sqrt{2}+1}{\sqrt{3}-\sqrt{2}+1}$$

$$=\frac{-2\sqrt{2}}{(\sqrt{3}-\sqrt{2})^2-1}=\frac{-2\sqrt{2}}{4-2\sqrt{6}},$$

and $\dfrac{-2\sqrt{2}}{4-2\sqrt{6}}$ or $\dfrac{2\sqrt{2}}{2\sqrt{6}-4}=\dfrac{2\sqrt{2}\,(2\sqrt{6}+4)}{(2\sqrt{6}-4)\,(2\sqrt{6}+4)}$

$$=\frac{4\sqrt{12}+8\sqrt{2}}{24-16}=\frac{8\sqrt{3}+8\sqrt{2}}{8}=\sqrt{3}+\sqrt{2}.$$

(4) $\dfrac{18+2\sqrt{6}}{4\sqrt{2}+2\sqrt{3}}=3\sqrt{2}-\sqrt{3}.$

(5) $\dfrac{3\sqrt{15}-4}{2\sqrt{3}-\sqrt{5}}=2\sqrt{5}+\sqrt{3}.$

(6) $\dfrac{2+2\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}=1-\sqrt{2}+\sqrt{3}.$

(7) $\left(\dfrac{a^3b^2}{cx^3}\right)^{\frac{1}{n}}\div\left(\dfrac{ab}{cx}\right)^{\frac{1}{n}}=\left(\dfrac{a^2b}{x^2}\right)^{\frac{1}{n}}.$

(8) $\sqrt{\{ab+\sqrt{(ac)}\}}\div\sqrt{a}=\sqrt{\left(b+\sqrt{\dfrac{c}{a}}\right)}.$

(9) $a\sqrt{\dfrac{bc+bx}{c}}\div b\sqrt{\dfrac{ad-ax}{d}}=\sqrt{\dfrac{ad(c+x)}{bc(d-x)}}.$

(10) $a\sqrt{x}+\sqrt{(bx)}-a\sqrt{y}-\sqrt{(by)}\div\sqrt{x}-\sqrt{y}=a+\sqrt{b}.$

(11) $\dfrac{(a^2-bc^2-x)\sqrt{x}-2cx\sqrt{b}}{a+c\sqrt{b}+\sqrt{x}}=a\sqrt{x}-c\sqrt{(bx)}-x.$

(12) $\left(\dfrac{a^4}{x}-a^2x+\dfrac{2a^2x^2}{\sqrt{(ax)}}+\dfrac{x^4}{a}\right)\div\left(\dfrac{a^2}{\sqrt{x}}+a\sqrt{x}+\dfrac{x^2}{\sqrt{a}}\right)=$
$\dfrac{a^2}{\sqrt{x}}-a\sqrt{x}+\dfrac{x^2}{\sqrt{a}}.$

105. The following are examples for practice in the reduction of compound expressions involving surds.

(1) $\sqrt{(ax)} + \dfrac{ax}{a - \sqrt{(ax)}} = \dfrac{a\sqrt{x}}{\sqrt{a} - \sqrt{x}}$.

(2) $\dfrac{1}{a + \sqrt{(a^2 - x^2)}} + \dfrac{1}{a - \sqrt{(a^2 - x^2)}} = \dfrac{2a}{x^2}$.

(3) $\dfrac{\sqrt{(1-x)} + \dfrac{1}{\sqrt{(1+x)}}}{1 + \dfrac{1}{\sqrt{(1-x^2)}}} = \sqrt{(1-x)}$.

(4) $\dfrac{\dfrac{1+x}{\sqrt{(1+x^2)}} - \dfrac{\sqrt{(1+x^2)}}{1+x}}{\dfrac{1+x}{\sqrt{(1+x^2)}} + \dfrac{\sqrt{(1+x^2)}}{1+x}} = \dfrac{x}{1+x+x^2}$.

(5) $\dfrac{\dfrac{\sqrt{(1-x)}}{2\sqrt{(1+x)}} + \dfrac{\sqrt{(1+x)}}{2\sqrt{(1-x)}}}{1-x} = \dfrac{1}{(1-x).\sqrt{(1-x^2)}}$

or $\dfrac{1}{(1-x)^{\frac{3}{2}}\sqrt{(1+x)}}$.

(6) $\dfrac{\sqrt{(x^2+x+1)} + \sqrt{(x^2-x-1)}}{\sqrt{(x^2+x+1)} - \sqrt{(x^2-x-1)}} =$

$$\frac{\{\sqrt{(x^2+x+1)} + \sqrt{(x^2-x-1)}\}^2}{x^2+x+1-(x^2-x-1)} = \frac{x^2 + \sqrt{\{x^4 - (x+1)^2\}}}{x+1}.$$

(7) $\dfrac{\sqrt{(1+x^2)} \mp x}{\sqrt{(1+x^2)} \pm x} = 1 + 2x^2 \mp 2x\sqrt{(1+x^2)}$.

(8) $\dfrac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}} = \dfrac{a + \sqrt{(a^2-x^2)}}{x}$

(9) $$\frac{1-x+2x^{\frac{3}{2}}+x^3}{1-2x^{\frac{1}{2}}+x+2x^{\frac{3}{2}}-2x^2+x^3}=\frac{1+x^{\frac{1}{2}}+x^{\frac{3}{2}}}{1-x^{\frac{1}{2}}+x^{\frac{3}{2}}}.$$

the common factor being $1-x^{\frac{1}{2}}+x^{\frac{3}{2}}$.

(10) $$\frac{a-x}{a^{\frac{5}{2}}-a^2x^{\frac{1}{2}}+a^{\frac{3}{2}}x-ax^{\frac{3}{2}}+a^{\frac{1}{2}}x^2-x^{\frac{5}{2}}}=\frac{\sqrt{a}+\sqrt{x}}{a^2+ax+x^2}.$$

(11) $$\frac{(x+4)\,x-2\,(x+3)\sqrt{x}+3}{(x-2)\,x-(x-3)\sqrt{x}-1}=\frac{x+3}{x+\sqrt{x}-1}.$$

(12) $$\frac{42x^{\frac{5}{2}}-43x^2y^{\frac{1}{2}}+36x^{\frac{3}{2}}y-5xy^{\frac{3}{2}}}{36x^{\frac{3}{2}}y-48xy^{\frac{3}{2}}+37x^{\frac{1}{2}}y^2-5y^{\frac{3}{2}}}=$$

$$\frac{x\,\{7x-6\sqrt{(xy)}+5y\}}{y\,\{6x-7\sqrt{(xy)}+5y\}}.$$

IMAGINARY QUANTITIES.

106. If a root be indicated by an even number, and the sign of the quantity be negative, the expression is called an *Impossible* or *Imaginary Quantity*, because no quantity raised to an even power can produce a negative result.

107. Every imaginary quantity may be shewn to involve $\sqrt{-1}$; thus,

$$\sqrt{-a^2}=\sqrt{\{a^2(-1)\}}=\pm a\sqrt{-1},\quad \sqrt{-b}=\sqrt{b}.\sqrt{-1},$$

&c.; the only ambiguity which can arise from treating them as surds, and which exists in the result of their multiplication, may therefore be removed by the simple consideration that involution and evolution are inverse operations: thus, instead of $(\sqrt{-a})^2=\sqrt{(-a)^2}=\sqrt{a^2}=\pm a$, we shall have

$(\sqrt{-a})^2 = -a$, the operation indicated by the radical sign being obviously neutralized by the inverse operation denoted by the index 2; and on the same principle we shall have

$$\sqrt{-a}.\sqrt{-b} = \sqrt{a}.\sqrt{-1}.\sqrt{b}.\sqrt{-1} = \sqrt{(ab)}\ (\sqrt{-1})^2 = -\sqrt{(ab)},$$

and not $\sqrt{-a}\,.\,\sqrt{-b} = \sqrt{\{(-a)\ (-b)\}} = \sqrt{(ab)}$.

108. Hence, in order to determine the sign of any product of imaginary quantities, we must observe, besides the signs $+$ and $-$ which may be prefixed to them, how often $\sqrt{-1}$ is involved: the investigation of the powers of $\sqrt{-1}$ becomes, therefore, a matter of importance.

$$(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -\sqrt{-1},$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2\,(\sqrt{-1})^2 = (-1)\,(-1) = 1:$$

and similarly,

$$(\sqrt{-1})^{4n} = \{(\sqrt{-1})^4\}^n = 1^n = 1,$$

$$(\sqrt{-1})^{4n+1} = (\sqrt{-1})^{4n}\,.\,\sqrt{-1} = \sqrt{-1},$$

$$(\sqrt{-1})^{4n+2} = (\sqrt{-1})^{4n}\,.\,(\sqrt{-1})^2 = -1,$$

$$(\sqrt{-1})^{4n+3} = (\sqrt{-1})^{4n+2}\,.\,\sqrt{-1} = -\sqrt{-1}.$$

In the last four are comprised all the powers of $\sqrt{-1}$; for all positive whole numbers are of one or other of the forms $4n$, $4n+1$, $4n+2$, $4n+3$, n being a whole number or zero: thus,

if $n=0$ these forms are 0, 1, 2, 3;

$n=1$ 4, 5, 6, 7;

$n=2$ 8, 9, 10, 11;

and so on.

EXAMPLES.

(1) To find the square of $a+b\sqrt{-1}$.

$$\begin{array}{l} a+b\sqrt{-1} \\ a+b\sqrt{-1} \\ \hline a^2+ab\sqrt{-1} \\ \qquad ab\sqrt{-1}+b^2(-1) \\ \hline a^2+2ab\sqrt{-1}-b^2 \end{array}$$

(2) To divide a^2+b^2 by $a+b\sqrt{-1}$.

$$\begin{array}{l} a+b\sqrt{-1})\ a^2+b^2\ (a-b\sqrt{-1} \\ \qquad\qquad a^2+ab\sqrt{-1} \\ \hline \qquad\qquad -ab\sqrt{-1}+b^2 \\ \qquad\qquad -ab\sqrt{-1}+b^2 \\ \hline \qquad\qquad \cdot \quad \cdot \end{array}$$

(3) To find the product of $2+3\sqrt{-1}$ and $3+\sqrt{-1}$.

$$\begin{array}{l} 2+3\sqrt{-1} \\ 3+\sqrt{-1} \\ \hline 6+9\sqrt{-1} \\ \quad 2\sqrt{-1}+3(-1) \\ \hline 6+11\sqrt{-1}-3=3+11\sqrt{-1} \end{array}$$

(4) $$\frac{3+11\sqrt{-1}}{2+3\sqrt{-1}}=\frac{3+11\sqrt{-1}}{2+3\sqrt{-1}}\times\frac{2-3\sqrt{-1}}{2-3\sqrt{-1}}$$

$$=\frac{39+13\sqrt{-1}}{4+9}=3+\sqrt{-1},$$ (See Ex. 2, Art. 104).

(5) $$\frac{a-b\sqrt{-1}}{x+y\sqrt{-1}}=\frac{a-b\sqrt{-1}}{x+y\sqrt{-1}}\times\frac{x-y\sqrt{-1}}{x-y\sqrt{-1}}$$

$$=\frac{ax-by}{x^2+y^2}-\frac{bx+ay}{x^2+y^2}\sqrt{-1}.$$

109. Subjoined are a few examples for practice in the transformation and reduction of expressions involving $\sqrt{-1}$.

(1) $\sqrt{-48}=\sqrt{(-16\times 3)}=4\,.\sqrt{3}\,.\sqrt{-1}$.

(2) $(a+b\sqrt{-1})\pm(c+d\sqrt{-1})=a\pm c+(b\pm d)\sqrt{-1}$.

(3) $(a\sqrt{-c})(b\sqrt{-d})=ab\sqrt{(cd)}\,(\sqrt{-1})^2=-ab\sqrt{(cd)}$.

(4) $\{a+\sqrt{(-4b^2)}\}\{a^2-3\sqrt{(-b^4)}\}=\{a+2b\sqrt{(-1)}\}\{a^2-3b^2\sqrt{(-1)}\}$
$=a^3+(2a^2b-3ab^2)\sqrt{-1}+6b^3$.

(5) $\{x+\sqrt{(-4y)}\}\{x-\sqrt{(-9y)}\}=x^2-x\sqrt{y}\sqrt{-1}+6y$.

(6) $(a+b\sqrt{-1})^3-(a-b\sqrt{-1})^3=(6a^2b-b^3)\sqrt{-1}$.

(7) $(a+b\sqrt{-1}+c\sqrt{-1})(a+b\sqrt{-1}-c\sqrt{-1})$
$=(a+b\sqrt{-1})^2+c^2=a^2-b^2+c^2+2ab\sqrt{-1}$.

(8) $\dfrac{a^2-b^2}{\sqrt{a}+\sqrt{b}\,.\sqrt{-1}}=a^{\frac{3}{2}}-a^{\frac{1}{2}}b-(ab^{\frac{1}{2}}-b^{\frac{3}{2}})\sqrt{-1}$.

(9) $\dfrac{a^3-b^3\sqrt{-1}}{a+b\sqrt{-1}}=a^2-ab\sqrt{-1}-b^2$.

(10) $(5+2\sqrt{-1})(6-3\sqrt{-1})=36-3\sqrt{-1}$.

(11) $\dfrac{17(1-\sqrt{-1})}{3-5\sqrt{-1}}=4+\sqrt{-1}$.

(12) $\left(\dfrac{1\pm\sqrt{-1}}{\sqrt{2}}\right)^4=-1$.

(13) $\dfrac{1}{a+b\sqrt{-1}}+\dfrac{1}{a-b\sqrt{-1}}=\dfrac{2a}{a^2+b^2}$.

(14) $\dfrac{a+b\sqrt{-1}}{c+d\sqrt{-1}}-\dfrac{a-b\sqrt{-1}}{c-d\sqrt{-1}}=\dfrac{2(bc-ad)\sqrt{-1}}{c^2+d^2}$.

EQUATIONS.

110. The term *Equation* is generally applied to any expressions, including zero, which are connected by the sign $=$.

111. In the theory of equations those only are considered which contain one or more unknown quantities; the object being to determine, from the relations expressed between the known and unknown quantities, the values of the latter in terms of the former.

These values are termed the *roots* of the equation, and are said to *satisfy* the equation, because, when substituted for the unknown quantities, they render its members identical.

112. The general principle by means of which a root of an equation may be discovered is, that any change may be made in its members which does not affect their equality. Hence,

(1) The same quantity may be added to, or subtracted from, both sides of an equation.

(2) Any term may be transferred from one side of an equation to the other, its sign being changed: for if a positive term be thus transferred, it is equivalent to subtracting the same quantity from both sides of the equation; and if a negative quantity be transferred, it is equivalent to adding the same quantity to both sides.

(3) Both members of an equation may be multiplied or divided by the same or equal quantities.

Hence, an equation may be cleared of fractions by multiplying both its members by each of the denominators, or by the least common multiple of all the denominators.

Also, the signs of all the terms on both sides may be changed, since this will only be introducing into each term the factor -1.

Hence also, both members of an equation may be raised to any power; and conversely, any root of both may be extracted.

113. When an equation is cleared of fractions and surds, if it contain the first power only of an unknown quantity, it is called a *simple* equation, or an equation of the *first* degree: if it contain the square of an unknown quantity, it is called a *quadratic*, or an equation of the *second* degree; and generally, if the index of the highest power of an unknown quantity in any equation be n, whether the inferior powers be involved or not, it is called an equation of the n^{th} degree.

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

114. Every simple equation containing but one unknown quantity may be put under the form

$$x - p = 0, \text{ or } x = p:$$

whence it is obvious that it can have but one root.

EXAMPLES.

(1) $$5x - 4 = 3x + 10$$

by transposition, $$5x - 3x = 10 + 4$$

$$\text{or, } 2x = 14$$

$$\therefore \quad x = 7.$$

(2) $$\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + 7$$

multiplying both sides by 12, the least com. multiple of 2, 3, and 4,

$$6x + 4x = 3x + 84$$

$$\therefore \text{ by transposition, } 6x + 4x - 3x = 84$$

$$\text{or, } 7x = 84$$

$$\therefore x = 12.$$

(3) $$\frac{x+5}{7} - \frac{x-2}{5} = \frac{x+9}{11}$$

multiplying by 7, 5, and 11,

$$55x + 275 - 77x + 154 = 35x + 315$$

transposing and collecting the quantities,

$$57x = 114$$

$$\therefore x = \frac{114}{57} = 2.$$

Obs. In clearing an equation of fractions, the sign $-$ affects every term in the numerator of the fraction to which it is prefixed.

(4) $$\sqrt{x} - 4 = \frac{259 - 10x}{\sqrt{x} + 4}$$

multiplying by $\sqrt{x} + 4$,

$$x - 16 = 259 - 10x$$

$$\therefore 11x = 259 + 16 = 275$$

$$\therefore x = \frac{275}{11} = 25.$$

(5) $$\sqrt{x - 16} + \sqrt{x} = 8$$

transposing $\sqrt{x}$ and squaring both sides,

$$x - 16 = 64 - 16\sqrt{x} + x$$

$$\therefore \quad \sqrt{x} = 5$$

$$\therefore \quad x = 25.$$

(6) $$\sqrt[3]{(2x+3)} + 4 = 7$$

transposing 4 and cubing both sides,

$$2x + 3 = 27$$

$$\therefore \quad 2x = 24$$

$$\therefore \quad x = 12.$$

(7) $$\frac{\sqrt{x} + 2a}{\sqrt{x} + b} = \frac{\sqrt{x} + 4a}{\sqrt{x} + 3b}$$

multiplying both sides of the equation by $(\sqrt{x}+b)(\sqrt{x}+3b)$,

$$x + (2a + 3b)\sqrt{x} + 6ab = x + (4a + b)\sqrt{x} + 4ab$$

$$(2a - 2b)\sqrt{x} = 2ab$$

$$\therefore \quad \sqrt{x} = \frac{ab}{a-b}$$

$$\therefore \quad x = \left(\frac{ab}{a-b}\right)^2.$$

(8) $$\frac{x - ax}{\sqrt{x}} = \frac{\sqrt{x}}{x}$$

multiplying both sides of the equation by $\sqrt{x}$,

$$x - ax = \frac{x}{x} = 1$$

$$\text{or,} \quad (1-a)x = 1$$

$$\therefore \quad x = \frac{1}{1-a}.$$

(9) $$\frac{3x - 1}{\sqrt{(3x)} + 1} = 1 + \frac{\sqrt{(3x)} - 1}{2}.$$

Since $3x - 1 = (\sqrt{3x} + 1)(\sqrt{3x} - 1)$,

$$\frac{3x - 1}{\sqrt{(3x)} + 1} = \sqrt{(3x)} - 1;$$

$$\therefore \quad \sqrt{(3x)} - 1 = 1 + \frac{\sqrt{(3x)} - 1}{2}$$

and subtracting $\frac{\sqrt{(3x)} - 1}{2}$ from each side,

$$\frac{\sqrt{(3x)} - 1}{2} = 1$$

$$\therefore \quad \sqrt{(3x)} - 1 = 2$$

$$\therefore \quad \sqrt{(3x)} = 3$$

$$\therefore \quad 3x = 9$$

$$\therefore \quad x = 3.$$

(10) $\frac{1}{x} + \frac{1}{a} = \sqrt{\left\{\frac{1}{a^2} + \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)}\right\}}$

squaring both sides,

$$\frac{1}{x^2} + \frac{2}{ax} + \frac{1}{a^2} = \frac{1}{a^2} + \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)}$$

$$\therefore \quad \frac{1}{x^2} + \frac{2}{ax} = \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)}$$

and multiplying both sides by x,

$$\frac{1}{x} + \frac{2}{a} = \sqrt{\left(\frac{4}{a^2} + \frac{9}{x^2}\right)}$$

again squaring both sides,

$$\frac{1}{x^2} + \frac{4}{ax} + \frac{4}{a^2} = \frac{4}{a^2} + \frac{9}{x^2}$$

$$\therefore \quad \frac{4}{ax} = \frac{8}{x^2}$$

and multiplying by x, $\frac{1}{a} = \frac{2}{x}$

$\therefore$ multiplying by ax, $x = 2a$.

(11) $$\sqrt{\left(\frac{a^2}{x}+b\right)}-\sqrt{\left(\frac{a^2}{x}-b\right)}=c$$

First, $\frac{a^2}{x}+b-2\sqrt{\left(\frac{a^4}{x^2}-b^2\right)}+\left(\frac{a^2}{x}-b\right)=c^2$

whence also, $4\left(\frac{a^4}{x^2}-b^2\right)=\frac{4a^4}{x^2}-\frac{4a^2c^2}{x}+c^4$

$$\therefore\ 4b^2=\frac{4a^2c^2}{x}-c^4$$

$$\therefore\ (4b^2+c^4)\,x=4a^2c^2$$

and $\therefore\ x=\frac{4a^2c^2}{4b^2+c^4}$.

(12) $$\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}=\sqrt[3]{b}$$

or $\sqrt[3]{(a+\sqrt{x})}=\sqrt[3]{b}-\sqrt[3]{(a-\sqrt{x})}$

$$\therefore\ a+\sqrt{x}=b-3b^{\frac{2}{3}}(a-\sqrt{x})^{\frac{1}{3}}+3b^{\frac{1}{3}}(a-\sqrt{x})^{\frac{2}{3}}-a+\sqrt{x}$$

$$\therefore\ 2a-b=-3b^{\frac{1}{3}}(a-\sqrt{x})^{\frac{1}{3}}\{b^{\frac{1}{3}}-(a-\sqrt{x})^{\frac{1}{3}}\}$$

$$=-3b^{\frac{1}{3}}(a-\sqrt{x})^{\frac{1}{3}}(a+\sqrt{x})^{\frac{1}{3}}$$

$$=-3b^{\frac{1}{3}}(a^2-x)^{\frac{1}{3}}$$

$$\therefore 8a^3-12a^2b+6ab^2-b^3=-27b(a^2-x)=-27a^2b+27bx$$

$$\therefore\ x=\frac{8a^3+15a^2b+6ab^2-b^3}{27b}.$$

EXAMPLES FOR PRACTICE.

(1) $12x-13=10x+15$ $\qquad x=14.$

(2) $3x+7=9x-5$ $\qquad x=2.$

(3) $ax\pm b=c$ $\qquad x=\frac{c\mp b}{a}.$

(4) $ax+a_1=bx+b_1$ $\qquad x=\dfrac{b_1-a_1}{a-b}$.

(5) $\dfrac{x}{2}+\dfrac{x}{3}=\dfrac{1}{4}$ $\qquad x=\dfrac{3}{10}$.

(6) $\dfrac{x}{2}-9=\dfrac{x}{6}+5$ $\qquad x=42$.

(7) $\dfrac{x}{2}+\dfrac{x}{3}-1=\dfrac{3x}{4}$ $\qquad x=12$.

(8) $x+\dfrac{x}{2}+\dfrac{x}{3}=137-\dfrac{x}{4}-\dfrac{x}{5}$ $\qquad x=60$.

(9) $\dfrac{b}{a}x-\dfrac{a}{b}=\dfrac{d}{c}-\dfrac{c}{d}x$ $\qquad x=\dfrac{a^2cd+abd^2}{abc^2+b^2cd}$.

(10) $\dfrac{x+1}{2}=\dfrac{3x+8}{7}$ $\qquad x=9$.

(11) $x=\dfrac{ax-b}{c}$ $\qquad x=\dfrac{b}{a-c}$.

(12) $\dfrac{x}{3}-\dfrac{7-x}{2}+14-x=0$ $\qquad x=63$.

(13) $x=\dfrac{4x}{7}+\dfrac{3(x-2)}{5}$ $\qquad x=7$.

(14) $x-\dfrac{x-1}{2}=3-\dfrac{3+x}{5}$ $\qquad x=2\frac{5}{7}$.

(15) $\dfrac{9-2x}{2}=1\frac{1}{2}-\dfrac{7x-18}{10}$ $\qquad x=4$.

(16) $x+\dfrac{2x-4}{3}=12-\dfrac{3x-5}{2}$ $\qquad x=5$.

(17) $\dfrac{x-1}{7}-\dfrac{x-23}{5}=7-\dfrac{4+x}{4}$ $\qquad x=8$.

(18) $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=2\frac{1}{3}$ $\qquad x=\dfrac{1}{2}$.

(19) $\dfrac{42}{1-2x}=\dfrac{35}{1-3x}$ $\qquad x=\dfrac{1}{8}$.

(20) $\dfrac{c}{a+bx}=\dfrac{f}{d+ex}$ $\qquad x=\dfrac{af-cd}{ce-bf}$.

(21) $\dfrac{2x-3}{3x-4}=\dfrac{4x-5}{6x-7}$ $\qquad x=1$.

(22) $\dfrac{60+8x}{x+3}-\dfrac{48}{x+1}=14-\dfrac{30+6x}{x+1}$ $\qquad x=3$.

(23) $\frac{1}{7}(x-\frac{1}{2})+\frac{1}{5}(x+\frac{2}{3})=1\frac{13}{30}$ $\qquad x=4$.

(24) $3.75x+.5=2.25x+8$ $\qquad x=5$.

(25) $.15x+1.2-.875x+.375=.0625x$ $\qquad x=2$.

(26) $8.4x-7.6=2.2x+10$ $\qquad x=2.8387\ldots$

(27) $a+x+\sqrt{(2ax+x^2)}=b$ $\qquad x=\dfrac{(a-b)^2}{2b}$.

(28) $\sqrt{12+x}=2+\sqrt{x}$ $\qquad x=4$.

(29) $\sqrt{4x+9}-2\sqrt{x}=1$ $\qquad x=4$.

(30) $\sqrt{x}=\sqrt{2}+\sqrt{(x-2)}$ $\qquad x=2$.

(31) $\sqrt{x-a}+\sqrt{x+b}=2\sqrt{x}$ $\qquad x=\dfrac{(a+b)^2}{8(b-a)}$.

(32) $\sqrt{4a+x}+\sqrt{a+x}=2\sqrt{2a+x}$ $\qquad x=-\dfrac{7a}{8}$.

(33) $\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}$ $\qquad x=\dfrac{2a}{5}$.

(34) $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}$ $\qquad x=\dfrac{a}{3}$.

(35) $\dfrac{1}{a}\sqrt{a+x}+\dfrac{1}{x}\sqrt{a+x}=\dfrac{1}{b}\sqrt{x}$ $\qquad x=\dfrac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}$.

(36) $3(\sqrt[3]{x}-6)=2(1-\sqrt[3]{x})$ $\qquad x=64$.

(37) $(a+x)^{\frac{1}{m}}=(x^2+5ax+b^2)^{\frac{1}{2m}}$ $\qquad x=\dfrac{a^2-b^2}{3a}$.

(38) $\dfrac{\sqrt{(a+\sqrt{x})}}{\sqrt[3]{x}}+\dfrac{\sqrt{(a-\sqrt{x})}}{\sqrt[3]{x}}=\sqrt[6]{x}$ $\qquad x=4(a-1)$.

(39) $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=b$ $\qquad x=\dfrac{2ab}{1+b^2}$.

(40) $\frac{1}{2}\sqrt{(\sqrt{-x}+3a^2)}-\frac{1}{2}\sqrt{(\sqrt{-x}-3a^2)}=\sqrt[4]{\left(-\frac{a^2}{b^2}x\right)}$

$$x=\frac{9a^3b^2}{4(a-b)}.$$

SIMPLE EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

115. When there are two independent simple equations involving two unknown quantities, the value of each unknown quantity may be obtained by any of the following methods.

First Method. In either equation, find the value of one of the unknown quantities in terms of the other and known quantities, and for it substitute this value in the other equation, which will then contain only one unknown quantity; and the value of this quantity being substituted in the expression for the first unknown quantity will give its value.

Thus, suppose $ax+by=c$ and $a'x+b'y=c'$; then, if we substitute for x in the second equation its value, $\dfrac{c-by}{a}$, determined from the first, the resulting equation will be

$$\frac{a'c-a'by}{a}+b'y=c'$$

$$\text{or} \quad a'c - a'by + ab'y = ac'$$

$$\therefore \ ab'y - a'by = ac' - a'c$$

$$\therefore \ y = \frac{ac' - a'c}{ab' - a'b}$$

and substituting the value of y in the expression for x, we obtain

$$x = \frac{c - by}{a} = \frac{c}{a} - \frac{b}{a} \cdot \frac{ac' - a'c}{ab' - a'b} = \frac{b'c - bc'}{ab' - a'b}.$$

Ex. 1. $3x + 2y = 118$ and $x + 5y = 191$.

From the first equation $x = \dfrac{118 - 2y}{3}$

and substituting this expression for x in the second equation,

$$\frac{118 - 2y}{3} + 5y = 191$$

$$\therefore \ 118 - 2y + 15y = 573$$

$$\therefore \ y = \frac{455}{13} = 35$$

hence also

$$x = \frac{118 - 2y}{3} = \frac{118 - 70}{3} = 16.$$

The simultaneous values of x and y are, therefore, 16 and 35, respectively; which, substituted in the original equations, will be found to satisfy them.

Ex. 2. $\dfrac{a}{x} - \dfrac{b}{y} = m$, and $\dfrac{c}{x} - \dfrac{d}{y} = n$.

Putting v for $\dfrac{1}{x}$ and w for $\dfrac{1}{y}$, we have

$$av - bw = m, \quad \text{and} \quad cv - dw = n;$$

and from these equations are obtained

$$v = \frac{bn - dm}{cb - ad}, \quad \text{and} \quad w = \frac{an - cm}{cb - ad};$$

whence $\quad x = \frac{1}{v} = \frac{cb - ad}{bn - dm}, \quad$ and $\quad y = \frac{1}{w} = \frac{cb - ad}{an - dm}.$

Second Method. Find from each equation an expression for one of the unknown quantities; put these expressions equal to each other, and the resulting equation will contain only one of the unknown quantities, the value of which may be found therefrom; and this, substituted in either of the above expressions, will give the value of the other unknown quantity. Thus, supposing

$$ax + by = c, \quad \text{and} \quad a'x + b'y = c'$$

then, from the first equation, $x = \frac{c - by}{a}$

and from the second, $x = \frac{c' - b'y}{a'}$

whence $\frac{c - by}{a} = \frac{c' - b'y}{a'}$

$$\therefore \; a'c - a'by = ac' - ab'y$$

and, by transposition, $ab'y - a'by = ac' - a'c$

or $(ab' - a'b)\,y = ac' - a'c$

$$\therefore \; y = \frac{ac' - a'c}{ab' - a'b}$$

and this value, being substituted in either of the above values of x, gives

$$x = \frac{b'c - bc'}{ab' - a'b}.$$

Ex. 3. $\frac{x+2}{3} + 8y = 31$ and $\frac{y+5}{4} + 10x = 192$.

From the first equation, $x + 2 + 24y = 93$

$$\therefore\ x = 91 - 24y \ldots\ldots(1)$$

from the second, $y + 5 + 40x = 768$

$$\therefore\ 40x = 763 - y$$

$$\therefore\ x = \frac{763 - y}{40} \ldots\ldots(2)$$

equating (1) and (2), $\frac{763 - y}{40} = 91 - 24y$

$$\therefore\ 763 - y = 3640 - 960y$$

$$\therefore\ 959y = 2877$$

$$\therefore\ y = \frac{2877}{959} = 3$$

and substituting the value of y in (1), we obtain

$$x = 91 - 72 = 19.$$

Third Method. Multiply or divide each of the given equations by such factors as will render the coefficients of one of the unknown quantities the same in both: then cancel the identical terms by addition or subtraction, and the result will be an equation containing but one unknown quantity.

Thus, if $ax + by = c$ and $a'x + b'y = c'$, and we multiply the former equation by a' and the latter by a, in order that the coefficient of x may be the same in both equations, we shall have

$$\left.\begin{aligned} aa'x + a'by &= a'c \\ \text{and } aa'x + ab'y &= ac' \end{aligned}\right\}$$

$\therefore$ by subtraction, $a'by - ab'y = a'c - ac'$

$$\text{whence } y = \frac{a'c - ac'}{a'b - ab'} \text{ or } \frac{ac' - a'c}{ab' - a'b}$$

and by substituting the value of y in either of the original equations,

$$\text{we shall obtain } x = \frac{bc' - b'c}{a'b - ab'} \text{ or } \frac{b'c - bc'}{ab' - a'b}, \text{ as before.}$$

Ex. 4. $\left.\begin{array}{r}(x + 5)(y + 7) = (x + 1)(y - 9) + 112 \\ \text{and } 2x + 10 = 3y + 1\end{array}\right\}$

Performing the multiplications indicated in the first equation,

$$xy + 7x + 5y + 35 = xy - 9x + y - 9 + 112$$

$$\therefore\ 16x + 4y = 68$$

$$\left.\begin{array}{r}\therefore\ 4x + \ y = 17 \\ \text{also, from the second equation, } 2x - 3y = -\ 9\end{array}\right\}$$

and multiplying the second of these equations by 2, and subtracting the product from the first, there results

$$7y = 35$$

$$\text{whence } y = 5$$

$$\text{also, } 2x = 3y - 9 = 15 - 9 = 6$$

$$\text{and } \therefore\ x = 3.$$

Ex. 5. $\dfrac{a}{b + y} = \dfrac{b}{3a + x}$ and $ax + 2by = c$.

From the first equation, $3a^2 + ax = b^2 + by$

$$\left.\begin{array}{l}\text{or } ax - by = b^2 - 3a^2 \\ \text{also } ax + 2by = c\end{array}\right\}$$

and since the coefficient of x is the same in both these equations, we shall have, by subtraction,

$$3by = 3a^2 - b^2 + c$$

$$\text{whence } y = \frac{3a^2 - b^2 + c}{3b}$$

$$\text{also } ax = c - 2by = c - \frac{6a^2 - 2b^2 + 2c}{3}$$

$$\text{and } \therefore x = \frac{c - 6a^2 + 2b^2}{3}.$$

Ex. 6.
$$\left.\begin{aligned} x + y &= a \\ \text{and } x - y &= b \end{aligned}\right\}$$

By addition $2x = a + b$, whence $x = \dfrac{a+b}{2}$

and by subtraction $2y = a - b$, whence $y = \dfrac{a-b}{2}$.

Ex. 7. $\dfrac{2x - y}{2} + 14 = 18$ and $\dfrac{x + 2y}{3} + 16 = 19$.

$$\left.\begin{aligned} \text{From the first equation } 2x - y &= 8 \\ \text{and from the second } 2x + 4y &= 18 \end{aligned}\right\}$$

$$\therefore \text{ by subtraction, } 5y = 10$$

$$\text{and } \therefore y = 2$$

$$\text{also } x = 9 - 2y = 9 - 4 = 5.$$

EXAMPLES FOR PRACTICE.

(1) $\dfrac{x + y}{3} + 2 = 7,$ $\qquad x = 11,$

$\dfrac{x - y}{7} + 3 = 4.$ $\qquad y = 4.$

(2) $\dfrac{x}{7} + 7y = 99,$ $\qquad x = 7,$

$\dfrac{y}{7} + 7x = 51.$ $\qquad y = 14.$

(3) $x + 5 = 3y - 9,$ $\qquad x = 25,$

$\frac{5x - 4}{11} - \frac{2x - 5}{9} = 17 - \frac{4y - 8}{4}.$ $\qquad y = 13.$

(4) $\frac{x + y}{3} = 1 - \frac{3x - 2y}{4},$ $\qquad x = 1,$

$17x - 31y = 1\frac{1}{2}$ $\qquad y = \frac{1}{2}.$

(5) $\frac{x}{4} + \frac{3y}{8} = 9,$ $\qquad x = 12,$

$\frac{x}{4} + \frac{y}{6} = 5\frac{2}{3}.$ $\qquad y = 16.$

(6) $ax + by = c,$ $\qquad x = \frac{bf + ce}{ae + bd},$

$dx - ey = f.$ $\qquad y = \frac{cd - af}{ae + bd}.$

(7) $\frac{x}{a} - \frac{y}{b} = m,$ $\qquad x = \frac{ac\,(bm + dn)}{ad + bc},$

$\frac{x}{c} + \frac{y}{d} = n.$ $\qquad y = \frac{bd\,(cn - am)}{ad + bc}.$

(8) $ax + by = c^2,$ $\qquad x = \frac{b^2 + c^2 - a^2}{2a},$

$\frac{a}{b + y} - \frac{b}{a + x} = 0.$ $\qquad y = \frac{a^2 + c^2 - b^2}{2b}.$

(9) $ax - by = c,$ $\qquad x = \frac{ac + 3a^2b - b^3}{a^2 - b^2},$

$\frac{a}{b - x} - \frac{b}{3a - y} = 0.$ $\qquad y = \frac{3a^3 + cb - ab^2}{a^2 - b^2}.$

(10) $\frac{6x + 9}{4} + \frac{3x + 5y}{4x - 6} = 3\frac{1}{4} + \frac{3x + 4}{2},$ $\qquad x = 7,$

$\frac{8y + 7}{10} + \frac{6x - 3y}{2y - 8} = 4 + \frac{4y - 9}{5}.$ $\qquad y = 9.$

116. If there be three independent simple equations, and three unknown quantities, from two of the equations obtain one containing only two of the unknown quantities, by any of the above methods, then from the third equation and either of the others obtain another containing the same two unknown quantities; and from the two equations thus obtained, the unknown quantities which they involve may be found. The third unknown quantity may be found by substituting the values of the two others in any of the given equations.

The same principle may also be extended to any number of independent simple equations containing as many unknown quantities.

EXAMPLES.

(1) $$5x - 6y + 4z = 15,$$
$$7x + 4y - 3z = 19,$$
and $$2x + y + 6z = 46.$$

From the first equation, $z = \dfrac{15 - 5x + 6y}{4}$;

$\therefore$ by substitution, $7x + 4y - \dfrac{45 - 15x + 18y}{4} = 19,$

and $2x + y + \dfrac{45 - 15x + 18y}{2} = 46$;

or $43x - 2y = 121,$

and $20y - 11x = 47$:

from the former of these equations, $y = \dfrac{43x - 121}{2}$,

$\therefore$ by substitution in the latter, $430x - 1210 - 11x = 47$;

whence $x = \dfrac{1257}{419} = 3,$

$$y = \frac{[illegible] - 12}{[illegible]} = \frac{[illegible] - 12}{[illegible]} = 4,$$

and $$z = \frac{[illegible] - 3x + [illegible]y}{4} = \frac{15 - 15 + 24}{4} = 6.$$

2.
$$2x + 3y - 4z = 8,$$
$$3x - 2y + 5z = 10,$$
and $$5x + 6y - 3z = 24.$$

From the first equation, $x = \dfrac{8 - 3y + 4z}{2}$

from the second, $x = \dfrac{10 + 2y - 5z}{3}$

and from the third, $x = \dfrac{24 - 6y + 3z}{5}$

equating the first and second values of x, we have

$$\frac{8 - 3y + 4z}{2} = \frac{10 + 2y - 5z}{3}$$

$$24 - 9y + 12z = 20 + 4y - 10z$$

$$y = \frac{22z + 4}{13}$$

equating the first and third,

$$\frac{8 - 3y + 4z}{2} = \frac{24 - 6y + 3z}{5}$$

$$40 - 15y + 20z = 48 - 12y + 6z$$

$$y = \frac{14z - 8}{3}$$

and equating these expressions for y,

$$\frac{22z + 4}{13} = \frac{14z - 8}{3}$$

$$66z + 12 = 182z - 104$$

$$\text{whence} \quad z = 1$$

$$\text{also} \quad y = \frac{14z - 8}{3} = 2$$

$$\text{and} \quad x = \frac{8 - 3y + 4z}{2} = \frac{8 - 6 + 4}{2} = 3.$$

Or thus:

$$\text{From the first two equations,} \quad \left.\begin{aligned} 6x + 9y - 12z &= 24 \\ \text{and} \quad 6x - 4y + 10z &= 20 \end{aligned}\right\}$$

$$\therefore \text{ by subtraction,} \quad 13y - 22z = 4$$

$$\text{from the first and third,} \quad \left.\begin{aligned} 10x + 15y - 20z &= 40 \\ \text{and} \quad 10x + 12y - 6z &= 48 \end{aligned}\right\}$$

$$\therefore \text{ by subtraction,} \quad \left.\begin{aligned} 3y \quad 14z &= -8 \\ \text{also} \quad 13y - 22z &= 4 \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{whence} \quad 39y - 182z &= -104 \\ \text{and} \quad 39y - 66z &= 12 \end{aligned}\right\}$$

$$\therefore \text{ by subtraction,} \quad -116z = -116$$

$$\therefore z = 1$$

$$\text{but,} \quad 3y = 14z - 8 = 6 \quad \therefore y = 2$$

$$\text{also} \quad 2x + 3y - 4z = 8 \quad \text{and} \therefore x = \frac{8 - 6 + 4}{2} = 3.$$

Or by the *First Method.*

(3)

$$z + y = a,$$

$$z + x = b,$$

$$\text{and} \quad y + x = c.$$

$$\text{From the first equation,} \quad z = a - y$$

$$\therefore \text{ by substitution in the second,} \quad a - y + x = b$$

$$\text{or} \quad y - x = a - b$$

$$\text{and} \quad y + x = c$$

$$\text{whence by subtraction,}\quad x = \frac{b + c - a}{2}$$

$$\text{by addition,}\quad y = \frac{a + c - b}{2}$$

$$\text{and}\quad z = b - x = \frac{a + b - c}{2}.$$

(4)
$$\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = 7\tfrac{3}{5},$$

$$\frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} = 10\tfrac{1}{6},$$

$$\text{and}\quad \frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} = 16\tfrac{1}{10}.$$

Putting u for $\frac{1}{x}$, v for $\frac{1}{y}$, and w for $\frac{1}{z}$, the equations become

$$3u - \tfrac{4}{5}v + w = \tfrac{38}{5},$$

$$\tfrac{1}{3}u + \tfrac{1}{2}v + 2w = \tfrac{61}{6},$$

$$\text{and}\quad \tfrac{4}{5}u - \tfrac{1}{2}v + 4w = \tfrac{161}{10}.$$

from which we obtain $u = 2$, $v = 3$, $w = 4$:

and $\therefore\ x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$.

(5)
$$xy = a(x + y),$$

$$xz = b(x + z),$$

$$\text{and}\quad yz = c(y + z).$$

Dividing the equations by axy, bxz, and cyz, respectively, they become

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a},$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{b},$$

$$\text{and}\quad \frac{1}{y} + \frac{1}{z} = \frac{1}{c}.$$

adding these together, and dividing by 2,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\text{whence,}\quad \frac{1}{x}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)$$

$$\frac{1}{y}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{c}-\frac{1}{b}\right)$$

$$\frac{1}{z}=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)$$

$$\text{and } \therefore\ x=\frac{2abc}{ac+bc-ab},$$

$$y=\frac{2abc}{ab+bc-ac},$$

$$\text{and}\quad z=\frac{2abc}{ab+ac-bc}.$$

EXAMPLES FOR PRACTICE.

(1) $2x + 4y - 3z = 22$, $\qquad x = 3$,
$4x - 2y + 5z = 18$, $\qquad y = 7$,
$6x + 7y - z = 63$. $\qquad z = 4$.

(2) $\dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{4}=62$, $\qquad x = 24$,
$\dfrac{x}{3}+\dfrac{y}{4}+\dfrac{z}{5}=47$, $\qquad y = 60$,
$\dfrac{x}{4}+\dfrac{y}{5}+\dfrac{z}{6}=38$. $\qquad z = 120$.

(3) $5x + 3y = 65$, $\qquad x = 7$,
$2y - z = 11$, $\qquad y = 10$,
$3x + 4z = 57$. $\qquad z = 9$.

(4) $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}$, $\qquad x=2$,

$\dfrac{5}{4x}+\dfrac{1}{2z}=\dfrac{3}{4}$, $\qquad y=3$,

$\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{7}{12}$. $\qquad z=4$.

(5) $a(x+c)=bz$, $\qquad x=\dfrac{ab^2c(c-a)-a^3c^2}{(a^3+b^3)c+a^2b^2}$,

$b(y+c)=az$, $\qquad y=\dfrac{a^2bc(c-b)-b^3c^2}{(a^3+b^3)c+a^2b^2}$,

$bcx+acy+abz=0$. $\qquad z=\dfrac{abc^2(a+b)}{(a^3+b^3)c+a^2b^2}$.

(6) $\dfrac{xy}{ay+bx}=l$, $\qquad x=\dfrac{(bde+acf)lmn}{cfmn-bfln+bdlm}$,

$\dfrac{yz}{cz+dy}=m$, $\qquad y=\dfrac{(bde+acf)lmn}{afln+demn-adlm}$,

$\dfrac{xz}{ez+fx}=n$. $\qquad z=\dfrac{(bde+acf)lmn}{bcln-cemn+aclm}$.

QUADRATIC EQUATIONS.

117. When an equation contains the square only of the unknown quantity, it is called a *Pure* or *Binomial Quadratic Equation;* and it is solved by finding the value of the square of the unknown quantity, as in simple equations, and extracting the root of both sides.

Thus, if $ax^2\pm b=0$, $x^2=\mp\dfrac{b}{a}$, and $\therefore x=\pm\sqrt{\mp\dfrac{b}{a}}$.

If $\dfrac{b}{a}$ be negative the solution of the equation gives an imaginary result, which will nevertheless be found, by

substitution, to satisfy the equation. It may also be observed that, since the solution of every quadratic equation involves the extraction of the square root, such equations must have two roots (Art. 84).

EXAMPLES.

(1) $$3x^2 - 4 = 71$$

$$\therefore\ x^2 = \frac{71 + 4}{3} = 25$$

$$\therefore\ x = \pm\ 5.$$

(2) $$\frac{a - \sqrt{(a^2 - x^2)}}{a + \sqrt{(a^2 - x^2)}} = b$$

Clearing of fractions, $a - \sqrt{(a^2 - x^2)} = ab + b\sqrt{(a^2 - x^2)}$

$$\therefore\ (b+1)\sqrt{(a^2 - x^2)} = a - ab = a(1-b)$$

$$\therefore\ \sqrt{(a^2 - x^2)} = a\,.\,\frac{1-b}{1+b}$$

$$\therefore\ a^2 - x^2 = a^2\left(\frac{1-b}{1+b}\right)^2$$

$$\therefore\ x^2 = a^2 - a^2\left(\frac{1-b}{1+b}\right)^2 = a^2.\frac{(1+b)^2 - (1-b)^2}{(1+b)^2} = a^2.\frac{4b}{(1+b)^2}$$

$$\therefore\ x = \pm\ \frac{2a.\sqrt{b}}{1+b}.$$

EXAMPLES FOR PRACTICE.

(1) $ax^2 - b = c.$ $\qquad x = \pm\sqrt{\frac{b+c}{a}}.$

(2) $4x + \frac{16}{x} = 0.$ $\qquad x = \pm\ 2\sqrt{-1}.$

(3) $\frac{a}{x} + \frac{\sqrt{(a^2 - x^2)}}{x} = \frac{x}{b}$ $\qquad x = \pm\ \sqrt{(2ab - b^2)}.$

(4) $\dfrac{2x \pm \sqrt{(4x^2-1)}}{2x \mp \sqrt{(4x^2-1)}} = 4.$ $\qquad x = \pm \frac{5}{8}.$

(5) $\sqrt{x^2+x} - \sqrt{x^2-x} = 2\sqrt{x^2-1}.$ $\qquad x = \pm \frac{2}{3}.$

118. Equations which contain both the first and second powers of an unknown quantity are called *Adfected Quadratics;* and are solved by reducing them to the form $x^2 \pm px = \pm q$, and then adding to both sides such a quantity as will complete the square of the first side, and extracting the root of both; thus obtaining a simple equation from which the value of the unknown quantity may be found.

119. *To complete the square in Quadratics.*

Supposing the equation reduced to the form

$$x^2 \pm px = \pm q,$$

let m be the quantity necessary to be added in order to make the first side a complete square: then observing that since $(a+b)^2 = a^2 + 2ab + b^2$, four times the product of the extreme terms = the square of the mean, in a trinomial which is a complete square, we shall have

$$4x^2m = p^2x^2, \text{ and } \therefore m = \frac{p^2}{4}:$$

from which it appears that the quantity necessary to be added is *the square of half the coefficient of the second term.*

120. Proceeding with the solution of the above equation, we shall have

$$x^2 \pm px + \left(\frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 \pm q$$

$$\therefore x \pm \frac{p}{2} = \pm \sqrt{\left\{\left(\frac{p}{2}\right)^2 \pm q\right\}}$$

$$\text{and } \therefore x = \mp \frac{p}{2} \pm \sqrt{\left\{\left(\frac{p}{2}\right)^2 \pm q\right\}}.$$

EXAMPLES.

(1) $$x^2 + 12x = 108.$$

Completing the square, $x^2 + 12x + 36 = 36 + 108 = 144$

and extracting the roots, $x + 6 = \pm 12$

$$\therefore\ x = \pm 12 - 6 = 6 \text{ or } -18.$$

(2) $$x^2 - x + 3 = 45$$

By transposition, $x^2 - x = 42$

completing the square, $x^2 - x + \frac{1}{4} = \frac{1}{4} + 42 = \frac{169}{4}$

and extracting the roots, $x - \frac{1}{2} = \pm \frac{13}{2}$

$$\therefore\ x = \frac{1}{2} \pm \frac{13}{2} = 7 \text{ or } -6$$

(3) $$ax^2 - bx = c$$

Dividing each term by a, $x^2 - \frac{b}{a}x = \frac{c}{a}$

completing the square,

$$x^2 - \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = \frac{b^2 + 4ac}{4a^2}$$

and extracting the roots, $x - \frac{b}{2a} = \pm \frac{\sqrt{(b^2 + 4ac)}}{2a}$

$$\therefore\ x = \frac{b \pm \sqrt{(b^2 + 4ac)}}{2a}.$$

(4) $$16 - \frac{2x^2}{3} = \frac{4x}{5} + 7\tfrac{3}{5}$$

By transposition, $\frac{2x^2}{3} + \frac{4x}{5} = 16 - 7\tfrac{3}{5}$

multiplying by $\frac{3}{2}$, $x^2 + \frac{6x}{5} = 24 - \frac{57}{5} = \frac{63}{5}$

completing the square, $x^2 + \frac{6x}{5} + \left(\frac{3}{5}\right)^2 = \frac{9}{25} + \frac{63}{5} = \frac{324}{25}$

and extracting the roots, $x + \frac{3}{5} = \pm \frac{18}{5}$

$$\therefore\ x = -\frac{3}{5} \pm \frac{18}{5} = 3 \text{ or } -\frac{21}{5}.$$

(5) $$\frac{10}{x} - \frac{14 - 2x}{x^2} = 2\tfrac{4}{9}$$

Multiplying by $9x^2$, $90x - 126 + 18x = 22x^2$

$$\therefore\ 22x^2 - 108x = -126$$

$$\therefore\ x^2 - \frac{54}{11}x = -\frac{63}{11}$$

$$\text{or } x^2 - \frac{54}{11}x + \left(\frac{27}{11}\right)^2 = \left(\frac{27}{11}\right)^2 - \frac{63}{11} = \frac{36}{(11)^2}$$

$$\therefore\ x - \frac{27}{11} = \pm \frac{6}{11}$$

$$\therefore\ x = \frac{27 \pm 6}{11} = 3 \text{ or } \frac{21}{11}.$$

(6) $$x + \frac{24}{x - 1} = 3x - 4$$

$$\text{or } \frac{12}{x - 1} = x - 2$$

$$\therefore\ 12 = (x - 1)(x - 2) = x^2 - 3x + 2$$

$$\therefore\ x^2 - 3x = 10$$

$$\text{or } x^2 - 3x + \frac{9}{4} = \frac{9}{4} + 10 = \frac{49}{4}$$

$$\therefore\ x - \frac{3}{2} = \pm \frac{7}{2}$$

$$\therefore\ x = \frac{3 \pm 7}{2} = 5 \text{ or } -2.$$

(7) $$\frac{x}{x+2} - \frac{x-9}{3x-20} = \frac{9}{13}$$

$$\text{First, } 13x - \frac{13x^2 - 91x - 234}{3x-20} = 9x + 18$$

$$\text{or } 4x - 18 = \frac{13x^2 - 91x - 234}{3x-20}$$

$$\text{and } 12x^2 - 134x + 360 = 13x^2 - 91x - 234$$

$$\text{or } x^2 + 43x = 360 + 234 = 594$$

$$\therefore\ x^2 + 43x + \left(\frac{43}{2}\right)^2 = 594 + \left(\frac{43}{2}\right)^2 = \frac{4225}{4}$$

$$\therefore\ x + \frac{43}{2} = \pm\sqrt{\frac{4225}{4}} = \pm\frac{65}{2}$$

$$\therefore\ x = \pm\frac{65}{2} - \frac{43}{2} = 11 \text{ or } -54.$$

(8) $$x^2 + 2ax + a^2 - 10x - 10a + 21 = 0$$

This equation may easily be solved by the usual method, but the following also is an easy solution.

$$(x+a)^2 - 10(x+a) = -21,$$

$$(x+a)^2 - 10(x+a) + 25 = 25 - 21 = 4,$$

$$\therefore\ x + a - 5 = \pm 2,$$

$$\text{and } \therefore\ x = 7 - a \text{ or } 3 - a.$$

(9) $$\frac{x}{\sqrt{x} + \sqrt{(a-x)}} + \frac{x}{\sqrt{x} - \sqrt{(a-x)}} = \frac{b}{\sqrt{x}}$$

Multiplying the equation by

$$\{\sqrt{x} + \sqrt{(a-x)}\}\{\sqrt{x} - \sqrt{(a-x)}\},$$

$$x\{\sqrt{x} - \sqrt{(a-x)} + \sqrt{x} + \sqrt{(a-x)}\} = \frac{b}{\sqrt{x}}(2x-a)$$

$$\text{or} \quad 2x\sqrt{x} = \frac{b}{\sqrt{x}}(2x - a)$$

$$\therefore x^2 - bx = -\frac{ab}{2}$$

$$\text{and} \quad x^2 - bx + \frac{b^2}{4} = \frac{b^2}{4} - \frac{ab}{2} = \frac{b^2 - 2ab}{4}$$

$$\therefore x = \frac{b \pm \sqrt{(b^2 - 2ab)}}{2}.$$

(10) $$x \pm \sqrt{(10x + 6)} = 9$$

By transposition, $\pm \sqrt{(10x + 6)} = 9 - x$

by involution, $10x + 6 = 81 - 18x + x^2$

$$\text{or} \quad x^2 - 28x = -75$$

$$\therefore x^2 - 28x + (14)^2 = 196 - 75 = 121$$

$$\therefore x - 14 = \pm 11$$

$$\text{and} \quad x = 14 \pm 11 = 25 \text{ or } 3.$$

Of these values it may be observed, that one belongs exclusively to the equation

$$x + \sqrt{(10x + 6)} = 9,$$

and the other to $x - \sqrt{(10x + 6)} = 9$,

whilst they both satisfy the rationalized equation, which involves the square of both

$$+\sqrt{(10x + 6)} \text{ and } -\sqrt{(10x + 6)}.$$

(11) $$x^2 - (a + b)x + ab = 0$$

$$\therefore x^2 - (a + b)x + \left(\frac{a+b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a-b}{2}\right)^2$$

$$\therefore x = \frac{a + b}{2} \pm \frac{a - b}{2} = a \text{ or } b.$$

EXAMPLES FOR PRACTICE.

(1) $x^2 - x = 2.$ $\qquad x = -1$, or 2.

(2) $3x^2 - 5x - 2 = 0.$ $\qquad x = 2$, or $-\frac{1}{3}$.

(3) $\frac{x^2}{3} + \frac{3x}{2} = 21.$ $\qquad x = 6$, or $-10\frac{1}{2}$.

(4) $x^2 - 7x = 11.$ $\qquad x = 8.32 \ldots$, or $1.32 \ldots$

(5) $(x - 5)(x + \frac{1}{5}) = 0.$ $\qquad x = 5$, or $-\frac{1}{5}$.

(6) $x^2 - (a - b)x - ab = 0.$ $\qquad x = a$, or $-b$.

(7) $(x + 2)^2 + (x + 2) = 20.$ $\qquad x = 2$, or -7.

(8) $x^2 - 4 = 16 - (x - 2)^2.$ $\qquad x = 4$, or -2.

(9) $ax^2 + bx - c = 0.$ $\qquad x = \frac{-b \pm \sqrt{(b^2 + 4ac)}}{2a}$.

(10) $\frac{1}{x} - 2x = x + \frac{1}{2}.$ $\qquad x = \frac{1}{2}$, or $-\frac{2}{3}$.

(11) $\frac{x}{a} + \frac{a}{x} = \frac{2}{a}.$ $\qquad x = 1 \pm \sqrt{(1 - a^2)}$.

(12) $\frac{4x}{x + 7} = \frac{x - 7}{2x + 3} + 2.$ $\qquad x = 7$, or $\frac{1}{3}$.

(13) $\frac{5x - 9}{x + 3} + \frac{8x + 44}{4x - 8} = 9.$ $\qquad x = 5$, or $-10\frac{1}{2}$.

(14) $\frac{1}{x+a} + \frac{1}{x+2a} + \frac{1}{x+3a} = \frac{3}{x}.$ $\qquad x = \frac{-11 \pm \sqrt{13}}{6}a$.

(15) $4x + 4\sqrt{x + 2} = 7.$ $\qquad x = 4\frac{1}{4}$, or $\frac{1}{4}$.

(16) $\sqrt{x + 5} \times \sqrt{x + 12} = 12.$ $\qquad x = 4$, or -21.

(17) $\sqrt{(4x + 5)} \cdot \sqrt{(7x + 1)} = 30.$ $\qquad x = 5$, or $-6\frac{11}{28}$.

(18) $\sqrt{x + 3} + \sqrt{x + 6} = 3\sqrt{x}.$ $\qquad x = \frac{9 \pm \sqrt{76}}{5}$.

(19) $\sqrt{5a+x}+\sqrt{5a-x}=\dfrac{12a}{\sqrt{5a+x}}$. $x=4a$, or $3a$.

(20) $x^2(x+4)+2x(x+4)=2-(x+4)$. $x=-2\pm\sqrt{3}$.

(21) $mqx^2-(mn-pq)x-np=0$. $x=\dfrac{n}{q}$, or $-\dfrac{p}{m}$.

(22) $\dfrac{x-\sqrt{(x+1)}}{x+\sqrt{(x+1)}}=\dfrac{5}{11}$. $x=8$, or $-\frac{8}{9}$.

(23) $\dfrac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=\dfrac{m}{n}$. $x=\pm\dfrac{a(\sqrt{m}\mp\sqrt{n})^2}{2\sqrt{mn}}$.

(24) $\dfrac{\sqrt{(a^2+ax+x^2)}-\sqrt{(a^2-ax+x^2)}}{\sqrt{(a^2+ax+x^2)}+\sqrt{(a^2-ax+x^2)}}=\dfrac{a}{b}$.

$$x=\frac{a^2+b^2}{4b}\pm\sqrt{\left\{\left(\frac{a^2+b^2}{4b}\right)^2-a^2\right\}}.$$

(25) $\dfrac{x^2}{\sqrt{a}+\sqrt{b}}-(\sqrt{a}-\sqrt{b})x=\dfrac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}$.

$$x=a, \text{ or } -b.$$

121. From Ex. 11 it appears that in a quadratic equation of the form $x^2\pm px\pm q=0$, the coefficient of the second term is the sum of the roots with their signs changed, and that the last term is the product of the roots.

It may also be observed, that the first side of the equation is the product $(x-a)(x-b)$, which, when $x=a$ or b, must $=0$; and whatever be the degree of an equation, it may be shewn to consist of a corresponding number of binomial factors, whose second terms are its respective roots, with their signs changed; and that consequently the number of roots corresponds with the exponent of the highest power of the unknown quantity.

122. If an equation involve two powers of an unknown quantity, and the exponent of one be double the exponent of the other, it may be solved as a quadratic equation.

EXAMPLES.

(1) $$x^4 - 6x^2 - 27 = 0$$

$$\text{or} \quad x^4 - 6x^2 = 27$$

$\therefore$ completing the square, $x^4 - 6x^2 + 9 = 9 + 27 = 36$

and taking the roots, $x^2 - 3 = \pm 6$

$$\therefore \; x^2 = 3 \pm 6 = 9, \text{ or } -3$$

which are the two values of x^2: hence, by evolution,

$$x = \pm 3, \text{ or } \pm \sqrt{-3}.$$

whence the four roots of the equation, or values of x, are

$$3, \; -3, \; +\sqrt{-3}, \text{ and } -\sqrt{-3}.$$

(2) $$3x^6 + 42x^3 - 3321 = 0$$

Transposing, dividing by 3, and completing the square,

$$x^6 + 14x^3 + 7^2 = 1107 + 49 = 1156$$

$$\therefore \; x^3 + 7 = \pm \sqrt{1156} = \pm 34$$

$$\therefore \; x^3 = -7 \pm 34 = 27, \text{ or } -41$$

$$\therefore \; x = 3, \text{ or } \sqrt[3]{-41};$$

which are only two of the six roots of the equation; but $1^{\frac{1}{3}}$ has, symbolically, the three values,

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \quad \text{and} \quad \frac{-1-\sqrt{-3}}{2},$$

which are the roots of the equation $x^3 - 1 = 0$; and $(-1)^{\frac{1}{3}}$ has the same three values with the contrary signs, being the roots of the equation $x^3+1=0$; the cube roots of 27 and -41, which are respectively equal to

$$3 \times 1^{\frac{1}{3}} \text{ and } (41)^{\frac{1}{3}} \times (-1)^{\frac{1}{3}},$$

have therefore each three different values; the former being

$$3, \quad -\tfrac{3}{2}(1-\sqrt{-3}), \text{ and } -\tfrac{3}{2}(1+\sqrt{-3}),$$

and the latter

$$-\sqrt[3]{41}, \quad \frac{\sqrt[3]{41}}{2}(1-\sqrt{-3}), \text{ and } \frac{\sqrt[3]{41}}{2}(1+\sqrt{-3});$$

which are the six roots of the equation.

(3) $$x^8 + 4x^4 = 12$$

Completing the square, $x^8 + 4x^4 + 4 = 12 + 4 = 16$

and extracting the roots, $x^4 + 2 = \pm 4$

$$\therefore\ x^4 = -2 \pm 4 = 2, \text{ or } -6$$

whence $x^2 = \pm\sqrt{2}$, or $\pm\sqrt{-6}$

and $\therefore\ x = \pm\sqrt{(\pm\sqrt{2})}$, or $\pm\sqrt{(\pm\sqrt{-6})}$;

which are the eight roots.

(4) $$x^{2n} - mx^n = p$$

Completing the square,

$$x^{2n} - mx^n + \frac{m^2}{4} = \frac{m^2}{4} + p = \frac{m^2 + 4p}{4}$$

and extracting the roots, $x^n - \dfrac{m}{2} = \pm\dfrac{\sqrt{(m^2 + 4p)}}{2}$

$$\therefore\ x^n = \frac{m \pm \sqrt{(m^2 + 4p)}}{2}$$

and $x = \left\{\dfrac{m \pm \sqrt{(m^2 + 4p)}}{2}\right\}^{\frac{1}{n}}$:

where it may be observed, that

$$\left\{\frac{m + \sqrt{(m^2 + 4p)}}{2}\right\}^{\frac{1}{n}} \text{ and } \left\{\frac{m - \sqrt{(m^2 + 4p)}}{2}\right\}^{\frac{1}{n}}$$

contain, respectively, n roots of the equation.

(5) $$x + 6x^{\frac{1}{2}} = 27$$

$$\therefore\ x + 6x^{\frac{1}{2}} + 9 = 9 + 27 = 36$$

$$\therefore\ \sqrt{x} = -3 \pm 6 = 3, \text{ or } -9$$

$$\text{and } \therefore\ x = 9, \text{ or } 81.$$

Both the values of $\sqrt{x}$ satisfy the original equation, but of the values of x, 9 alone satisfies it; 81, which is introduced by involution, belonging to the equation $x - 6x^{\frac{1}{2}} = 27$.

(6) $$x^3 - 6x^{\frac{3}{2}} = 16$$

Putting y for $x^{\frac{3}{2}}$, and completing the square, we shall have

$$y^2 - 6y + 9 = 25$$

$$\text{and } \therefore\ y \text{ or } x^{\frac{3}{2}} = 3 \pm 5 = 8, \text{ or } -2$$

$$\text{whence } \quad x = 4 \times 1^{\frac{1}{3}} \text{ or } 4^{\frac{1}{3}};$$

of which the three values of $4^{\frac{1}{3}}$ are introduced by involution, being the roots of the equation $x^3 + 6x^{\frac{3}{2}} + 16$.

(7) $$3x^n \sqrt[3]{x^n} + \frac{4x^n}{\sqrt[3]{x^n}} = 4$$

$$\text{or } \quad x^{\frac{4n}{3}} + \tfrac{4}{3}x^{\frac{2n}{3}} = \tfrac{4}{3}$$

$$\therefore\ x^{\frac{4n}{3}} + \tfrac{4}{3}x^{\frac{2n}{3}} + \tfrac{4}{9} = \tfrac{4}{9} + \tfrac{4}{3} = \tfrac{16}{9}$$

$$\therefore\ x^{\frac{2n}{3}} = \pm\tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}, \text{ or } -2$$

$$\text{and } \therefore\ x = (\tfrac{2}{3})^{\frac{3}{2n}}, \text{ or } (-2)^{\frac{3}{2n}}.$$

(8)
$$x - 1 = 2 + \frac{2}{\sqrt{x}}$$

$$\text{or} \quad (\sqrt{x} + 1)(\sqrt{x} - 1) = 2\,\frac{\sqrt{x} + 1}{\sqrt{x}}$$

$$\therefore \sqrt{x} - 1 = \frac{2}{\sqrt{x}}$$

$$\text{or} \quad x - \sqrt{x} = 2$$

$$\text{whence} \quad \sqrt{x} = 2, \text{ or } -1$$

$$\text{and } \therefore x = 4, \text{ or } 1;$$

of which values the latter is introduced by involution, being the root of the equation $x - 1 = 2 - \frac{2}{\sqrt{x}}$.

(9)
$$x - \frac{12 + 8\sqrt{x}}{x - 5} = 0$$

$$\text{or} \quad x^2 - 5x - 12 - 8\sqrt{x} = 0$$

by transposition, $x^2 - 4x = x + 8\sqrt{x} + 12$

and completing the square,

$$x^2 - 4x + 4 = x + 8\sqrt{x} + 16$$

each side of which equation is a complete square;

$$\text{whence} \quad x - 2 = \sqrt{x} + 4$$

$$\therefore x - \sqrt{x} + \tfrac{1}{4} = 6 + \tfrac{1}{4} = \tfrac{25}{4}$$

$$\therefore \sqrt{x} = \tfrac{1}{2} \pm \tfrac{5}{2} = 3, \text{ or } -2$$

$$\text{and } \therefore x = 9, \text{ or } 4:$$

of which values the latter is introduced by involution, being the root of the equation $x - \frac{12 - 8\sqrt{x}}{x - 5} = 0$.

(10) $$\frac{x}{x+4}+\frac{4}{\sqrt{(x+4)}}=\frac{21}{x}$$

$$\text{or } \frac{1}{x+4}+\frac{4}{x}\sqrt{\frac{1}{x+4}}=\frac{21}{x^2}$$

$$\therefore \frac{1}{x+4}+\frac{4}{x}\sqrt{\frac{1}{x+4}}+\frac{4}{x^2}=\frac{25}{x^2}$$

$$\therefore \sqrt{\frac{1}{x+4}}=\pm\frac{5}{x}-\frac{2}{x}=\frac{3}{x}, \text{ or } -\frac{7}{x}$$

$$\therefore \frac{1}{x+4}=\frac{9}{x^2}, \text{ or } \frac{49}{x^2}$$

$$\therefore x^2-9x=36, \text{ or } x^2-49x=196$$

whence $x=12$ or -3, or $\dfrac{49\pm\sqrt{3185}}{2}$; the positive and negative values corresponding to the like values of $\sqrt{(x+4)}$ in the original equation.

(11) $$x^4-2x^3+x=132$$

Adding and subtracting x^2, there results

$$x^4-2x+x^2-(x^2-x)=132$$

and completing the square,

$$(x^2-x)^2-(x^2-x)+\tfrac{1}{4}=132+\tfrac{1}{4}=\tfrac{529}{4}$$

$$\therefore x^2-x=\tfrac{1}{2}\pm\tfrac{23}{2}=12, \text{ or } -11$$

whence, by the solution of the equations $x^2-x=12$ and $x^2-x=-11$, we obtain

$$x=4 \text{ or } -3, \text{ and } \frac{1\pm\sqrt{-43}}{2}.$$

(12) $$x^4+4x^3+3x^2+4x+1=0$$

Dividing by x^2, the equation becomes

$$x^2+4x+3+\frac{4}{x}+\frac{1}{x^2}=0$$

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$$\text{or} \quad x^2 + \frac{1}{x^2} + 4\left(x + \frac{1}{x}\right) + 3 = 0$$

and observing that $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$

we obtain $\left(x + \frac{1}{x}\right)^2 + 4\left(x + \frac{1}{x}\right) + 1 = 0$

which is in the form of a quadratic, and gives

$$x + \frac{1}{x} = -2 \pm \sqrt{3}$$

and multiplying by x, $\quad x^2 + 1 = (-2 \pm \sqrt{3})\,x$

a quadratic equation, from which we obtain

$$x = \frac{-2 \pm \sqrt{3} \pm \sqrt{(3 \mp 4\sqrt{3})}}{2}.$$

(13) $\sqrt[3]{(a+x)^2} - 5\sqrt[3]{(a^2-x^2)} = -4\sqrt[3]{(a-x)^2}$

or $(a+x)^{\frac{2}{3}} - 5(a^2-x^2)^{\frac{1}{3}} = -4(a-x)^{\frac{2}{3}}$

Divide by $(a-x)^{\frac{2}{3}}$, then

$$\left(\frac{a+x}{a-x}\right)^{\frac{2}{3}} - 5\left(\frac{a+x}{a-x}\right)^{\frac{1}{3}} = -4$$

whence $\left(\frac{a+x}{a-x}\right)^{\frac{1}{3}} = \frac{5}{2} \pm \frac{3}{2} = 4, \text{ or } 1$

$$\therefore\ a + x = 64(a-x), \text{ or } a - x$$

and $\therefore\ x = \frac{63a}{65}$, or 0.

(14) $\sqrt{\frac{x}{x+a}}\sqrt[3]{\left(\frac{x}{x+a}\right)^{\frac{1}{2}}} - \sqrt{\frac{x}{x+a}}\sqrt[3]{\left(\frac{x+a}{x}\right)^{\frac{1}{2}}} = 6$

or $\left(\frac{x}{x+a}\right)^{\frac{3}{6}}\left(\frac{x}{x+a}\right)^{\frac{1}{6}} - \left(\frac{x}{x+a}\right)^{\frac{3}{6}}\left(\frac{x+a}{x}\right)^{\frac{1}{6}} = 6$

by reduction and completing the square,

$$\left(\frac{x}{x+a}\right)^{\frac{2}{3}} - \left(\frac{x}{x+a}\right)^{\frac{1}{3}} + \frac{1}{4} = \frac{1}{4} + 6 = \frac{25}{4}$$

$$\therefore \left(\frac{x}{x+a}\right)^{\frac{1}{3}} = \frac{1}{2} \pm \frac{5}{2} = 3, \text{ or } -2$$

$$\text{and } \therefore \frac{x}{x+a} = 27, \text{ or } -8$$

$$\text{whence } x = \frac{27a}{26}, \text{ or } -\frac{8a}{9}.$$

EXAMPLES FOR PRACTICE.

(1) $x^4-8x^2=9$. $x=\pm 3$, or $\pm\sqrt{-1}$.

(2) $x^{2n}-2x^n=3$. $x=3^{\frac{1}{n}}$, or $(-1)^{\frac{1}{n}}$.

(3) $x^n-2ax^{\frac{n}{2}}-b^2=0$. $x=\{a\pm\sqrt{(a^2+b^2)}\}^{\frac{2}{n}}$.

(4) $x+5=\sqrt{(x+5)}+6$. $x=4$, or -1.

(5) $x-\sqrt{x-1}=7$. $x=10$, or 5.

(6) $x^2-5x-19=\sqrt{(x^2-5x+1)}$. $x=8$, or -3.

(7) $ax=b+\sqrt{cx}$. $x=\dfrac{(c\pm\sqrt{c+4ab})^2}{4a^2}$.

(8) $\dfrac{49x^2}{4}-49+\dfrac{48}{x^2}=9+\dfrac{6}{x}$.

$x=2$, $-\dfrac{8}{7}$, or $\dfrac{-3\pm\sqrt{93}}{7}$.

(9) $\sqrt{x}+2=\sqrt{7+2x}$. $x=9$, or 1.

(10) $x^2=x^4-2x\sqrt{(x^2-1)}+1$. $x=\pm\sqrt{\dfrac{1\pm\sqrt{5}}{2}}$.

(11) $x^4 - x^3 + \frac{5x^2}{4} - x + 1 = 0$.

$$x = \frac{1 \pm 2 \pm \sqrt{\{(1 \pm 2)^2 - 16\}}}{4}.$$

(12) $x + 4 + \sqrt{\frac{x+4}{x-4}} = \frac{12}{x-4}$. $\qquad x = \pm 5$.

(13) $\sqrt[5]{(\sqrt{a} + \sqrt{x})^2} + \sqrt[5]{(\sqrt{a} - \sqrt{x})^2} = 6\sqrt[5]{(a-x)}$,

$$x = \left\{\frac{(3 \pm \sqrt{8})^5 - 1}{(3 \pm \sqrt{8})^5 + 1}\right\}^2 . a.$$

(14) $\sqrt{\frac{x-a}{x}} \sqrt[3]{\left(\frac{x-a}{x}\right)^{\frac{1}{4}}} - \sqrt[4]{\frac{x-a}{x}} \sqrt[3]{\left(\frac{x}{x-a}\right)^{\frac{1}{4}}} = 6$.

$$x = -\frac{a}{728}, \text{ or } -\frac{a}{63}.$$

(15) $(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{2}{m}} = 6\,(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{2}{m}} + (a - x)^{\frac{1}{m}}$.

$$x = \frac{a\{(1 \pm 5)^m - 2^m\}^2}{\{(1 \pm 5)^m + 2^m\}^2}.$$

123. When there are more equations and unknown quantities than one, a single equation, involving only one of the unknown quantities, may sometimes be obtained by the methods employed in simple equations; and one of the unknown quantities being discovered, the others may be obtained from the preceding equations, by substitution.

(1) $\qquad \frac{10x + y}{xy} = 3$ and $y - x = 2$

From the second equation, $y = x + 2$

from the first, $10x + y = 3xy$

and substituting in this, the above expression for y, we obtain

$$10x + x + 2 = 3x\,(x + 2)$$

or $\quad 3x^2 - 5x = 2$

a quadratic equation, which gives

$$x = 2, \text{ or } -\tfrac{1}{3},$$

$$\text{and} \quad y = x + 2 = 4, \text{ or } \tfrac{5}{3}.$$

(2) $$ax + by = c \text{ and } exy = f$$

From the first equation, $y = \dfrac{c - ax}{b}$

and substituting this in the second, we have

$$ex\,\frac{c - ax}{b} = f, \text{ or } x^2 - \frac{c}{a}x = -\frac{bf}{ae}$$

a quadratic equation, which gives

$$x = \frac{c}{2a} \pm \frac{1}{2ae}\sqrt{(c^2e^2 - 4abef)}$$

$$\text{whence} \quad y = \frac{c}{2b} \mp \frac{1}{2be}\sqrt{(c^2e^2 - 4abef)}.$$

(3) $$x^2 + y^2 = a \text{ and } xy = b$$

$$\left.\begin{aligned} \text{or} \quad x^2 + y^2 &= a \\ \text{and} \quad 2xy &= 2b \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore \text{ by addition,} \quad x^2 + 2xy + y^2 &= a + 2b \\ \text{by subtraction,} \quad x^2 - 2xy + y^2 &= a - 2b \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{and by evolution,} \quad x + y &= \pm\sqrt{(a + 2b)} \\ \text{and} \quad x - y &= \pm\sqrt{(a - 2b)} \end{aligned}\right\}$$

$$\text{whence} \quad x = \pm\tfrac{1}{2}\{\sqrt{(a + 2b)} \pm \sqrt{(a - 2b)}\},$$

$$\text{and} \quad y = \pm\tfrac{1}{2}\{\sqrt{(a + 2b)} \mp \sqrt{(a - 2b)}\}.$$

(4) $$4xy = 96 - x^2y^2 \text{ and } x + y = 6$$

From the first equation, $x^2y^2 + 4xy = 96$

$$\text{or} \quad x^2y^2 + 4xy + 4 = 100$$

$$\text{whence} \quad xy = 8, \text{ or } -12$$

now, squaring the second equation,

$$\left.\begin{array}{rl} & x^2+2xy+y^2=36 \\ \text{and from above} & 4xy \qquad\quad =32, \text{ or } -48 \end{array}\right\}$$

$$\therefore \text{ by subtraction, } \quad x^2-2xy+y^2=4, \text{ or } 84$$

$$\left.\begin{array}{rl} \text{and by evolution,} & x-y=\pm 2, \text{ or } \pm\sqrt{84} \\ \text{also} & x+y=6 \end{array}\right\}$$

$$\text{whence} \quad x=4,\ 2,\ 3\pm\sqrt{21},$$

$$\text{and} \quad y=2,\ 4,\ 3\mp\sqrt{21};$$

where it may be observed that x and y interchange values, that is, when $x=4$, $y=2$, and when $x=2$, $y=4$, &c., because they are similarly involved in the given equations.

(5) $$\frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9}, \text{ and } x-y=2$$

From the first equation,

$$\frac{x^2}{y^2}+\frac{4x}{y}+4=\frac{85}{9}+4=\frac{121}{9}$$

$$\therefore \frac{x}{y}=\frac{5}{3}, \quad \text{or } -\frac{17}{3}$$

$$\therefore x=\frac{5y}{3}, \quad \text{or } -\frac{17y}{3},$$

the former of which gives $y=x-2=3$,

$$\text{and } \therefore x=\frac{5y}{3}=5;$$

and the latter $y=x-2=-\dfrac{3}{10}$,

$$\text{and } \therefore x=-\frac{17y}{3}=\frac{17}{10}.$$

(6) $$ax+by=c \text{ and } a'x^2+b'y^2=c'$$

From the first equation, $y = \dfrac{c - ax}{b}$

and substituting this for y in the second equation, we have

$$a'x^2 + b'\left(\frac{c - ax}{b}\right)^2 = c'$$

a quadratic equation, from which we obtain

$$x = \frac{ab'c \pm b\sqrt{(a'b^2c' - a'b'c^2 + a^2b'c')}}{a'b^2 + a^2b'};$$

$$\text{and} \quad y = \frac{c}{b} - \frac{a}{b}x = \frac{a'bc \mp a\sqrt{(a'b^2c' - a'b'c^2 + a^2b'c')}}{a'b^2 + a^2b'}.$$

(7) $x + y - \dfrac{\sqrt{(x+y)}}{\sqrt{(x-y)}} = \dfrac{6}{x-y}$ and $x^2 + y^2 = 41$

Multiplying the first equation by $x - y$, we get

$$x^2 - y^2 - \sqrt{(x^2 - y^2)} = 6$$

which, solved as a quadratic, gives

$$\sqrt{(x^2 - y^2)} = \tfrac{1}{2} \pm \tfrac{5}{2} = 3, \text{ or } -2$$

$$\left.\begin{aligned} \text{and } \therefore\ x^2 - y^2 &= 9, \text{ or } 4 \\ \text{also } x^2 + y^2 &= 41 \end{aligned}\right\}$$

whence, by addition and subtraction,

$$2x^2 = 50, \text{ or } 45, \text{ and } 2y^2 = 32, \text{ or } 37$$

$$\text{and } \therefore\ x = \pm 5, \text{ or } \pm\sqrt{\frac{45}{2}}, \text{ and } y = \pm 4, \text{ or } \pm\sqrt{\frac{37}{2}}.$$

(8) $x^3 + y^3 = a$ and $x + y = b$

$$\left.\begin{aligned} \text{Dividing the first equation by the second, } x^2 - xy + y^2 &= \frac{a}{b} \\ \text{and squaring the second, } x^2 + 2xy + y^2 &= b^2 \end{aligned}\right\}$$

$$\therefore \text{ by subtraction, } \quad 3xy = b^2 - \frac{a}{b} = \frac{b^3 - a}{b}$$

$$\text{and } \therefore\ xy = \frac{b^3 - a}{3b}$$

$$\text{hence}\quad x^2 - 2xy + y^2 = \frac{a}{b} - \frac{b^3 - a}{3b} = \frac{4a - b^3}{3b}$$

$$\left.\begin{aligned} &\text{and } \therefore\ x - y = \pm\sqrt{\frac{4a - b^3}{3b}} \\ &\text{also}\quad x + y = b \end{aligned}\right\}$$

$$\text{and } \therefore\ x = \frac{b}{2} \pm \frac{1}{2}\sqrt{\frac{4a - b^3}{3b}}, \text{ and } y = \frac{b}{2} \mp \frac{1}{2}\sqrt{\frac{4a - b^3}{3b}}.$$

Or thus:

Cubing the second equation, $x^3 + 3x^2y + 3xy^2 + y^3 = b^3$

$$\text{or}\quad a + 3xy\,(x + y) = b^3$$

$$\text{whence}\quad xy = \frac{b^3 - a}{3b}$$

$$\left.\begin{aligned} &\therefore\ 4xy = \frac{4}{3b}(b^3 - a) \\ &\text{and squaring the second equation, } x^2 + 2xy + y^2 = b^2 \end{aligned}\right\}$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = b^2 - \frac{4}{3b}(b^3 - a) = \frac{4a - b^3}{3b}$$

$$\left.\begin{aligned} &\text{hence}\quad x - y = \pm\sqrt{\frac{4a - b^3}{3b}} \\ &\text{also}\quad x + y = b \end{aligned}\right\}$$

whence x and y, as before.

(9) $x + y = a$ and $x^4 + y^4 = b$

Since $x^4 + y^4$ is not divisible by $x + y$, without a remainder, raise the first equation to the fourth power, then

$$\left.\begin{aligned} x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 &= a^4 \\ \text{subtract the second,}\qquad x^4 + y^4 &= b \end{aligned}\right\}$$

$$\text{and there results}\quad 4x^3y + 6x^2y^2 + 4xy^3 = a^4 - b$$

$$\left.\begin{array}{r}\text{now, dividing by } xy,\ 4x^2+6xy+4y^2=\dfrac{a^4-b}{xy}\\ \text{and observing that } 4(x+y)^2=4x^2+8xy+4y^2=4a^2\end{array}\right\}$$

$$\text{we have, by subtraction,}\quad 2xy = 4a^2 - \frac{a^4-b}{xy}$$

$$\text{or}\quad x^2y^2 - 2a^2xy = \frac{b-a^4}{2}$$

an equation which, solved as a quadratic, gives

$$xy = a^2 \pm \sqrt{\frac{b+a^4}{2}} = m, \text{ suppose, and } \therefore\ x = \frac{m}{y}$$

and substituting this expression for x in the first equation, we obtain

$$\frac{m}{y} + y = a, \quad \text{or } y^2 - ay = -m$$

a quadratic equation, which gives

$$y = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - m\right)} = \frac{a}{2} \pm \sqrt{\left(-\frac{3a^2}{4} \mp \sqrt{\frac{b+a^4}{2}}\right)},$$

$$\text{and}\quad x = a - y = \frac{a}{2} \mp \sqrt{\left(-\frac{3a^2}{4} \mp \sqrt{\frac{b+a^4}{2}}\right)}.$$

When the unknown quantities are similarly involved, the sum and difference of two others may be substituted for them. If this artifice be employed in the above example, and we assume $x=v+z$ and $y=v-z$, we shall have

$$x^4 = v^4 + 4v^3z + 6v^2z^2 + 4vz^3 + z^4$$

$$y^4 = v^4 - 4v^3z + 6v^2z^2 - 4vz^3 + z^4$$

$$\therefore \text{ by addition,}\quad x^4 + y^4 = 2v^4 + 12v^2z^2 + 2z^4 = b$$

$$\text{or}\quad z^4 + 6v^2z^2 = \frac{b}{2} - v^4$$

whence $z = \pm \sqrt{\left\{-3v^2 \pm \sqrt{\left(\frac{b}{2} + 8v^4\right)}\right\}}$

and $\therefore\ x = v \pm \sqrt{\left\{-3v^2 \pm \sqrt{\left(\frac{b}{2} + 8v^4\right)}\right\}}$

and $y = v \mp \sqrt{\left\{-3v^2 \pm \sqrt{\left(\frac{b}{2} + 8v^4\right)}\right\}}$

or, substituting for v its value $\frac{1}{2}(x + y)$, or $\frac{a}{2}$,

$$x = \frac{a}{2} \pm \sqrt{\left(-\frac{3a^2}{4} \pm \sqrt{\frac{b + a^4}{2}}\right)},$$

$$\text{and}\quad y = \frac{a}{2} \mp \sqrt{\left(-\frac{3a^2}{4} \pm \sqrt{\frac{b + a^4}{2}}\right)}.$$

(10) $$\left.\begin{aligned} x^2 + xy &= a \\ \text{and}\quad y^2 + xy &= b \end{aligned}\right\}$$

By addition, $x^2 + 2xy + y^2 = a + b$

$$\therefore\ x + y = \pm \sqrt{(a + b)}$$

$$\text{hence}\quad x = \frac{a}{x + y} = \pm \frac{a}{\sqrt{(a + b)}},$$

$$\text{and}\quad y = \frac{b}{x + y} = \pm \frac{b}{\sqrt{(a + b)}}.$$

Or thus :

Assuming $y = vx$, and dividing the first equation by the second,

$$\frac{x^2 + vx^2}{v^2x^2 + vx^2} = \frac{a}{b}, \quad \text{or} \quad \frac{1 + v}{v(1 + v)} = \frac{a}{b}$$

$$\therefore\ v = \frac{b}{a} \text{ and } y = \frac{b}{a}x$$

$$\text{whence}\quad x^2 + \frac{b}{a}x^2 = a, \quad \text{or } x = \pm \frac{a}{\sqrt{(a + b)}},$$

$$\text{and} \quad y = \frac{b}{a}x = \pm \frac{b}{\sqrt{(a+b)}}.$$

The artifice employed in the second solution of this example, that is, of assuming one of the unknown quantities equal to the product of the other and a third unknown quantity, is frequently applied when the sum of the dimensions of the unknown quantities, in every term of each equation, is the same.

(11) $xz = y^2$, $x + y + z = 21$, and $x^2 + y^2 + z^2 = 189$

From the first and third equations, $x^2 + z^2 + 2xz = 189 + y^2$

$$\therefore \; x + z = \pm \sqrt{(189 + y^2)} = 21 - y$$

$$\therefore \; 189 + y^2 = 441 - 42y + y^2$$

$$\therefore \; y = 6$$

$$\left.\begin{array}{ll} \therefore \text{ from the third equation,} & x^2 + z^2 = 189 - y^2 = 153 \\ \text{and from the first,} & 2xz \;\;\;\;\; = 72 \end{array}\right\}$$

$$\therefore \text{ by subtraction} \quad x^2 - 2xz + z^2 = 81$$

$$\left.\begin{array}{ll} \therefore & x - z = \pm 9 \\ \text{also} & x + z = 21 - y = 15 \end{array}\right\}$$

$$\text{whence} \quad x = 12 \text{ or } 3, \text{ and } z = 3 \text{ or } 12.$$

Hence the corresponding values of x, y, z are 12, 6, 3 or 3, 6, 12, x and z being similarly involved in each of the given equations.

EXAMPLES FOR PRACTICE.

(1) $x - y = 6$, $\quad x = 9$, or -3,

$xy = 27$. $\quad y = 3$, or -9.

(2) $x + y = a$, $\quad x = \frac{1}{2}\{a \pm \sqrt{(a^2 - 4b)}\}$,

$xy = b$. $\quad y = \frac{1}{2}\{a \mp \sqrt{(a^2 - 4b)}\}$.

(3) $x^2 + y^2 = 34,$ $\qquad x = 5, \text{ or } -3,$

$x - y = 2.$ $\qquad y = 3, \text{ or } -5.$

(4) $x^2 + y^2 = a,$ $\qquad x = \frac{1}{2}(b \pm \sqrt{2a - b^2}),$

$x + y = b,$ $\qquad y = \frac{1}{2}(b \mp \sqrt{2a - b^2}).$

(5) $\frac{2}{x} + \frac{3}{y} = 8,$ $\qquad x = \frac{2}{7}, \text{ or } 2,$

$7xy = 6.$ $\qquad y = 3, \text{ or } \frac{3}{7}.$

(6) $\frac{x}{y} + y = 8,$ $\qquad x = 15, \text{ or } 16,$

$\frac{x}{y} + 1 = \frac{20}{y}.$ $\qquad y = 5, \text{ or } 4.$

(7) $\frac{1}{x} + \frac{1}{y} = m,$ $\qquad x = \frac{2}{m \pm \sqrt{(2n^2 - m^2)}},$

$\frac{1}{x^2} + \frac{1}{y^2} = n^2.$ $\qquad y = \frac{2}{m \mp \sqrt{(2n^2 - m^2)}}.$

(8) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4,$ $\qquad x = 9, \text{ or } 1,$

$x^{\frac{3}{2}} + y^{\frac{3}{2}} = 28.$ $\qquad y = 1, \text{ or } 9.$

(9) $x^2y + xy^2 = 180,$ $\qquad x = 5, \text{ or } 4,$

$x^3 + y^3 = 189.$ $\qquad y = 4, \text{ or } 5.$

(10) $x - y = 2,$ $\qquad x = 5, \text{ or } -3,$

$x^3 - y^3 = 98.$ $\qquad y = 3, \text{ or } -5.$

(11) $x + y = 5,$ $\qquad x = 2, 3, \frac{1}{2}(5 \pm \sqrt{-51}),$

$x^4 + y^4 = 97.$ $\qquad y = 3, 2, \frac{1}{2}(5 \mp \sqrt{-51}).$

(12) $x + y = 5,$ $\qquad x = 1, 4, \frac{1}{2}(5 \pm \sqrt{-59}),$

$x^5 + y^5 = 1025.$ $\qquad y = 4, 1, \frac{1}{2}(5 \mp \sqrt{-59}).$

(13) $x^2 - xy = 6,$ $\qquad x = \pm 6,\ \pm \frac{1}{\sqrt{2}},$

$x^2 + y^2 = 61.$ $\qquad y = \pm 5,\ \pm \frac{11}{\sqrt{2}}.$

(14) $\frac{x^2+y^2}{10} = \frac{x+y}{3},$ $\qquad x=4,\ 2,\ \frac{-4\pm 2\sqrt{-14}}{3},$

$xy = 8.$ $\qquad y=2,\ 4,\ \frac{-4\mp 2\sqrt{-14}}{3}.$

(15) $x+y+\sqrt{(x+y)}=6,$ $\qquad x=3,\ 1,\ \frac{9\pm\sqrt{-61}}{2},$

$x^2+y^2=10.$ $\qquad y=1,\ 3,\ \frac{9\mp\sqrt{-61}}{2}.$

(16) $x^2 + y^2 = 2a\,(x + y),$

$xy = ab.$

$$x=\tfrac{1}{2}\left\{a\pm\sqrt{a^2+2ab}\pm\sqrt{(a\pm\sqrt{a^2+2ab})^2-4ab}\right\},$$

$$y=\tfrac{1}{2}\left\{a\pm\sqrt{a^2+2ab}\mp\sqrt{(a\pm\sqrt{a^2+2ab})^2-4ab}\right\}.$$

(17) $x + y + \sqrt{(xy)} = 19,$ $\qquad x = 9,$ or $4,$

$x^2 + y^2 + xy = 133.$ $\qquad y = 4,$ or $9.$

(18) $x^2y^2 + 6xy = 16,$ $\qquad x = 1,\ -2,\ \frac{-1\pm\sqrt{-31}}{2},$

$\sqrt{\frac{1}{4x}}=\frac{1}{2}\sqrt{\frac{1}{y-1}}.$ $\qquad y = 2,\ -1,\ \frac{1\pm\sqrt{-31}}{2}.$

(19) $xz = y^2,$ $\qquad x = 8,$ or $2,$

$x + y + z = 14,$ $\qquad y = 4,$

$x^2 + y^2 + z^2 = 84.$ $\qquad z = 2,$ or $8.$

(20) $x(y+z)=a,\qquad x=\pm\sqrt{\dfrac{(a+c-b)(a+b-c)}{2(b+c-a)}},$

$y(x+z)=b,\qquad y=\pm\sqrt{\dfrac{(b+c-a)(a+b-c)}{2(a+c-b)}},$

$z(x+y)=c.\qquad z=\pm\sqrt{\dfrac{(b+c-a)(a+c-b)}{2(a+b-c)}}.$

PROBLEMS.

124. The solution of problems, by means of Algebra, is effected by expressing in algebraical language the quantities which they involve, together with the conditions to which they are subject, and reducing them, in conformity with such conditions, to an equation, or equations, the solution of which will give the *quæsita*, or things sought or required to be determined, in terms of the *data*, or things given.

EXAMPLES.

(1) Divide £128 among three persons, so that the first may have three times as much as the second, and the third one-third as much as the first and second together.

Let $x =$ the share of the second,

then $3x =$ ... ... first,

and $\dfrac{4x}{3} =$ ... ... third;

and since $128 =$ the whole sum,

$$x + 3x + \frac{4x}{3} = 128$$

whence $x = 24$, and $\therefore\ 3x = 72$, and $\dfrac{4x}{3} = 32$.

(2) A person distributed p shillings among n persons, giving 9 pence to some, and 15 pence to the rest. How many of each were there?

$$\text{Let } x = \text{the number that received } 15d.$$
$$\text{then } n - x = \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ 9d.$$
$$\therefore \ 15x + 9(n - x) = 12p$$
$$\text{whence } x = \frac{4p - 3n}{2}, \quad \text{and } \therefore n - x = \frac{5n - 4p}{2}.$$

(3) *A*, who travels $6\frac{1}{2}$ miles an hour, starts $2\frac{1}{2}$ hours before *B*, who travels the same road at the rate of 9 miles an hour. How long will *A* travel before he is overtaken by *B*?

Let x be the required time, in hours,

then $\frac{13x}{2}$ will be the distance *A* travels;

and by the question,

$$\frac{13x}{2} = 9\left(x - \frac{5}{2}\right)$$

whence $x = 9$ hours.

(4) At the review of an army, the troops were drawn up in a solid mass, 40 deep; when there were just one-fourth as many men in front as there were spectators. Had the depth, however, been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Determine the number of men of which the army consisted.

$$\text{Let } x = \text{the number in front,}$$
$$\text{then } 40x = \ \ldots \ \ldots \ \text{in the mass,}$$
$$\text{and } 4x = \ \ldots \ \ldots \ \text{of spectators.}$$

Hence the whole number present was $44x$; and by the question,

$$(x - 100)\,45 = 44x$$
$$\therefore \ x = 4500 \text{ men,}$$
$$\text{and } 40x = 180000, \text{ the number required.}$$

(5) *A* entered into a canal speculation with fourteen others, and the profits amounted in all to £595 more than five times the original price of a share. Seven of his partners in this affair joined him in a scheme for navigating the said canals with steam-boats, each venturing a sum of money less than his former gains by £173. But the steam-boats blowing up, *A* found he had lost £419 by them, for the company not only never recovered the money advanced, but lost all they had gained by digging the canals, and £368 besides. What were the prices of shares in the two concerns originally?

Let $x =$ the price of a share in the canal speculation,

$$\text{then the profits} = 5x + 595£$$

$$\therefore \text{ the gain of each} = \frac{5x + 595}{15} = \frac{x + 119}{3}$$

Again, each ventured on the steam-boats

$$\frac{x + 119}{3} - 173 = \frac{x - 400}{3}$$

$$\therefore \text{ the total advance} = \frac{8x - 3200}{3}$$

$$\text{and the total loss} = \frac{8x - 3200}{3} + \frac{8x + 952}{3} + 368$$

$$= \frac{16x - 1144}{3}$$

$$\therefore A\text{'s loss} = \frac{2x - 143}{3} = 419, \text{ by the question,}$$

$$\text{whence } x = 700£$$

and $\therefore$ the price of a share in the steam-boat speculation was

$$\frac{x - 400}{3} = 100£.$$

(6) A certain number consisting of two digits is equal to four times the sum of those digits; and if 12 be subtracted from twice the number, the digits will be inverted. What is the number?

Let x and y be the digits of tens and units respectively;

then $10x + y =$ the number itself,

$$\left.\begin{array}{r}\text{and by the question,}\quad 10x + y = 4(x + y)\\ \text{and}\quad 2(10x + y) - 12 = 10y + x\end{array}\right\}$$

whence $x = 4$, and $y = 8$,

$\therefore$ the number is 48.

(7) A person bought some wine for £60, and if he had bought 3 dozen more for the same sum, he would have paid £1 less per dozen. How many dozen did he buy?

Let $x =$ the number required,

then $\frac{60}{x} =$ the price per dozen,

and $\frac{60}{x+3} =$ the price per dozen on the second supposition;

and by the question, $\frac{60}{x+3} = \frac{60}{x} - 1$,

a quadratic equation, which gives

$x = 12$, or -15.

Of these values of x the former alone can fulfil the conditions of the problem, for there is no such abstract number as -15, (Art 5): but if in the equation $\frac{60}{x+3} = \frac{60}{x} - 1$, the negative sign be affixed to x, to denote diminution of stock instead of increase, the result will be 15 or -12, 15 alone answering the reversed conditions of the problem, which may be stated thus:

A person *sold* some wine for £60, and if he had sold three dozen *less* for the same sum, he would have *received* £1 *more* per dozen. How many dozen did he *sell?*

(8) Of two numbers one is equal to the square root of 16 times the other, and the sum of their squares is 225. Required the numbers.

$$\text{Let } x = \text{the second number,}$$
$$\text{then } \sqrt{(16x)} = \text{the first;}$$
$$\text{and by the question, } x^2 + 16x = 225,$$
$$\text{whence } x = 9, \text{ or } -25;$$

but as the latter value of x makes the first number an imaginary quantity, 9 alone answers the conditions of the problem, and therefore the numbers are 9 and 12.

(9) A body of men were formed into a hollow square, 3 deep, when it was observed that with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.

Let x = the number of men on one side of the hollow square,

$$\text{then } x^2-(x-6)^2 = \text{the whole number of men,}$$
$$\text{and } x^2-(x-6)^2+25=(x^{\frac{1}{2}}+22)^2$$
$$\text{or } x-4x^{\frac{1}{2}}=45$$
$$\text{whence } x^{\frac{1}{2}}=9, \text{ or } -5$$
$$\text{and } \therefore x=81, \text{ or } 25,$$

the latter of which is introduced by involution,

$$\therefore \text{ the whole number of men} = 936.$$

(10) The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased by one yard, it will make only four revolutions more than the hind-wheel in going that distance. Required the circumference of each.

Let $x =$ the number of yards in the circumference of the larger, and $y =$ the number in the circumference of the smaller;

$$\text{then } \frac{120}{x} = \frac{120}{y} - 6$$

$$\text{whence } xy = 20x - 20y$$

$$\text{and } \frac{120}{x+1} = \frac{120}{y+1} - 4$$

$$\text{whence } xy = 29x - 31y - 1 \text{ also,}$$

$$\therefore\ 29x - 31y - 1 = 20x - 20y$$

$$\text{and } x = \frac{11y + 1}{9}$$

$$\therefore \text{ by substitution, } \frac{11y^2 + y}{9} = \frac{220y + 20}{9} - 20y$$

$$\text{whence } y = 4, \text{ or } -\frac{5}{11}:$$

and therefore the number of yards in the circumference of the smaller is 4, and of the larger is $\frac{11y + 1}{9} = 5$.

EXAMPLES FOR PRACTICE.

(1) How many trees are there in an orchard containing $\frac{1}{5}$th pear trees, $\frac{3}{7}$ths apple trees, and 26 trees of other kinds?

Ans. 70.

(2) *A* and *B* start in trade with equal sums of money; *A* gains £126, and *B* loses £87, and now *A*'s money is twice *B*'s. What sum had each at first? *Ans.* £300.

(3) A and C can mow a field in m days, B and C in n days; in what time can the three do it together, supposing that A can mow r times as much as B in the same time?

Ans. $\dfrac{rn-m}{r-1}$ days.

(4) A garrison was victualled for 30 days; after 10 days it was reinforced by 3000 men, and then their provisions were exhausted in 5 days. Required the original number of the garrison. *Ans.* 1000.

(5) A sets out from Cambridge to London ($52\frac{1}{2}$ miles) at the rate of 8 miles an hour, and B sets out at the same time from London to Cambridge at the rate of $9\frac{1}{2}$ miles an hour; at what distance from Cambridge will they meet?

Ans. 24 miles.

(6) Two couriers pass through a place at an interval of 4 hours, travelling at the rates of $11\frac{1}{2}$ and $17\frac{1}{2}$ miles per hour; how far will the first travel before he is overtaken by the second? *Ans.* $134\frac{1}{6}$ miles.

(7) The garrison of a certain town consists of 1250 men, partly infantry and partly cavalry. Each horse-soldier receives £5 per month, and each foot-soldier £3. If the monthly pay of the garrison amount to £4150, how many horse and how many foot-soldiers does it contain?

Ans. 200 cavalry, and 1050 infantry.

(8) Find a number consisting of two digits, such, that when it is divided by the difference of its digits, the quotient is 21; and when it is divided by the sum of its digits, and the quotient increased by 17, the digits are inverted.

Ans. 42.

(9) A detachment was marching in regular column with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in 5 lines. Required the number of men. *Ans.* 4550.

(10) What two numbers are those, the difference of which multiplied by the greater produces 40, and by the less 15? *Ans.* 8 and 3.

(11) Find two numbers, the sum of which is 15, and the sum of their squares 113. *Ans.* 7 and 8.

(12) A and B can do a piece of work in m days, A and C in n days, B and C in r days: what portion of the work can each do in a day?

Ans. The portions which A, B, and C respectively can do in a day, are represented by $\frac{nr+mr-mn}{2mnr}$, $\frac{nr-mr+mn}{2mnr}$, and $\frac{mr-nr+mn}{2mnr}$.

INEQUALITIES.

125. An *Inequality* is an expression consisting of two members, with the sign $>$ or $<$ between them.

126. Both members of an inequality may be increased or diminished by any quantity, multiplied or divided by any quantity which does not change their signs, raised to any power, or any root of each extracted, and the same kind of inequality will evidently still subsist. (Art. 40.)

127. But if a factor be introduced which changes the sign of each member, the inequality will be reversed; for, if $A > B$, then, by transposition, $-B > -A$, or $-A < -B$.

EXAMPLES.

(1) To find whether one fraction $\frac{a}{b}$ be $>$, $=$, or $<$ another $\frac{c}{d}$.

Reducing them to the same denominator, they become

$$\frac{ad}{bd} \text{ and } \frac{bc}{bd};$$

whence it is obvious that $\frac{a}{b}$ is $>$, $=$, or $<$ $\frac{c}{d}$

according as ad is $>$, $=$, or $< bc$.

(2) If $\frac{a}{b}$ be any fraction whatever, then will $\frac{a}{b} + \frac{b}{a}$ be greater than 2.

For since $(a - b)^2$ is necessarily a positive quantity, whether a be greater or less than b,

$$a^2 - 2ab + b^2 \text{ is } > 0$$

and $\therefore$ by adding $2ab$ to both sides, we have

$$a^2 + b^2 > 2ab$$

$$\text{whence } \frac{a^2 + b^2}{ab} \text{ or } \frac{a}{b} + \frac{b}{a} \text{ is } > 2:$$

that is, *the sum of any quantity, except unit, and its reciprocal, is* > 2.

EXAMPLES FOR PRACTICE.

(1) Which is the greater, $\sqrt{2} + \sqrt{7}$ or $\sqrt{3} + \sqrt{5}$?

Ans. $\sqrt{2} + \sqrt{7}$.

(2) Which is the greater, $\frac{a - x}{a + x}$ or $\frac{a^2 - x^2}{a^2 + x^2}$?

Ans. $\frac{a^2 - x^2}{a^2 + x^2}$.

(3) Which is the greater, $\frac{a}{b^2} + \frac{b}{a^2}$ or $\frac{1}{b} + \frac{1}{a}$?

Ans. $\frac{a}{b^2} + \frac{b}{a^2}$.

RATIOS.

128. *Ratio* is the relation which one quantity bears to another, with respect to magnitude, the comparison being made by considering what multiple, part or parts, one is of the other.

129. The quantities to be compared must evidently be of the same kind; for although we can compare the abstract numbers 3 and 9, yet no comparison can exist between 3 pounds and 9 yards.

They must also be referred to the same unit; thus, in order to consider what part 5*s.* is of £3, we reduce these quantities to the same denomination, as 5*s.* and 60*s.* or £$\frac{1}{4}$ and £3.

130. If a and b be two magnitudes, their ratio will be represented by the fraction $\frac{a}{b}$; it is also written thus, $a : b$, and read thus, a to b; and the former term is called the *antecedent* of the ratio, and the latter the *consequent.*

131. *If the terms of a ratio be multiplied or divided by the same quantity, the ratio is not altered.*

For $\frac{a}{b} = \frac{ma}{mb}$ (Art. 68), or $a : b = ma : mb$.

132. A ratio is said to be a ratio of *greater* or *less inequality*, according as the antecedent is greater or less than the consequent.

133. *A ratio of greater inequality is diminished, and of less inequality increased, by adding any quantity to both its terms.*

Let x be added to the terms of the ratio $\frac{a}{b}$;

$$\text{then will } \frac{a}{b} \text{ be} > \text{or} < \frac{a+x}{b+x}$$

$$\text{according as } \frac{ab+ax}{b(b+x)} \text{ is} > \text{or} < \frac{ab+bx}{b(b+x)}$$

$$\text{or as } \quad ax \text{ is} > \text{or} < bx$$

$$\text{or as } \quad a \text{ is} > \text{or} > b;$$

that is, according as the original ratio $\frac{a}{b}$ is one of greater or less inequality.

134. Hence, conversely, *a ratio of greater inequality is increased, and of less inequality diminished, by subtracting from its terms a quantity less than either of them.*

135. If the antecedents of any ratios be multiplied together, and also the consequents, the resulting ratio is called their *sum*, or the ratio *compounded* of them.

Thus, the ratio $\frac{ace\ldots}{bdf\ldots}$ is the sum of the ratios $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, &c.

136. When the ratio $\frac{a}{b}$ is compounded with itself any number of times, the resulting ratio $\frac{a^2}{b^2}$, $\frac{a^3}{b^3}$, &c. is called the *duplicate, triplicate*, &c. ratio of $a : b$, or its *double, treble*, &c. ratio.

Also, the ratios $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$, $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$, &c. are termed the *sub-duplicate,*

sub-triplicate, &c. ratios of $a : b$, or the *half, third*, &c. ratios of $a : b$.

The indices, shewing what multiple or part of the simple ratio is taken, are called the *measures* of the ratios.

137. *If the consequent of the preceding ratio be the antecedent of the succeeding one, and any number of such ratios be taken, the ratio which results from their composition, is that of the first antecedent to the last consequent.*

Let $a : b$, $b : c$, $c : d$ be the ratios, the compound ratio is $abc : bcd$, or, dividing by bc, $a : d$.

138. *A ratio of greater inequality, compounded with another, increases it; and a ratio of less inequality diminishes it.*

For, let the ratio $x : y$ be compounded with the ratio $a : b$; then, the resulting ratio $ax : by$ is $>$ or $<$ $a : b$, according as $\frac{ax}{by}$ is $>$ or $<$ $\frac{a}{b}$, that is, according as x is $>$ or $< y$.

PROPORTION.

139. *Proportion* consists in the equality of ratios. Thus, if $\frac{a}{b} = \frac{c}{d}$, the four quantities a, b, c, d constitute a proportion, and are called proportionals, the first being the same multiple, part or parts, of the second, that the third is of the fourth.

This proportion is also written thus, $a : b = c : d$, but commonly thus, $a : b :: c : d$, and read thus, the ratio of a to b is equal to that of c to d, or thus, a is to b as c to d.

The terms a and d are called the *extremes*, and b and c the *means*.

140. ***When four quantities are proportionals, the product of the extremes is equal to the product of the means.***

For if $\frac{a}{b} = \frac{c}{d}$, we have, by multiplying both by bd, $ad = bc$.

141. If $a : b :: b : c$, in which case the three quantities a, b, c, are said to be in *continued* proportion, and b a *mean proportional* between a and c; then $ac = b^2$, or the *product of the extremes = the square of the mean.*

142. ***If the product of two quantities be equal to the product of two others, the four are proportionals, making the terms of one product the means, and the terms of the other the extremes.***

For, if $ad = bc$, then, dividing by bd, $\frac{a}{b} = \frac{c}{d}$,

$$\text{or} \quad a : b :: c : d.$$

143. Any three terms of a proportion being given the fourth may be determined from the equation $ad = bc$; thus, $a = \frac{bc}{d}$, $b = \frac{ad}{c}$, $c = \frac{ad}{b}$ and $d = \frac{bc}{a}$.

144. By means of the fractions which represent the proportion $a : b :: c : d$, it will be easy to shew that various other arrangements and modifications of its terms may be made, in which proportionality will still be preserved. Of these the following are the most useful.

If $a : b :: c : d$, then

(1) $a : c :: b : d$ (*Alternando*)

For, since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$, or $\frac{a}{c} = \frac{b}{d}$,

that is, $a : c :: b : d$.

(2) $b : a :: d : c$ (*Invertendo*)

For, since $\frac{a}{b} = \frac{c}{d}$, $\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d}$, or $\frac{b}{a} = \frac{d}{c}$

that is, $b : a :: d : c$.

(3) $a + b : b :: c + d : d$ (*Componendo*)

For, since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$,

that is, $a + b : b :: c + d : d$.

(4) $a - b : b :: c - d : d$ (*Dividendo*)

For, since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$, or $\frac{a-b}{b} = \frac{c-d}{d}$,

that is, $a - b : b :: c - d : d$.

(5) $a+b : a-b :: c+d : c-d$ (*Componendo & Dividendo*)

For, componendo, $\frac{a+b}{b} = \frac{c+d}{d}$,

and dividendo, $\frac{a-b}{b} = \frac{c-d}{d}$,

$\therefore$ by dividing the former of these by the latter, we have

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$, or $a + b : a - b :: c + d : c - d$.

(6) $a - b : a :: c - d : c$ (*Convertendo*)

For, invertendo,

$\frac{b}{a} = \frac{d}{c}$, $\therefore 1 - \frac{b}{a} = 1 - \frac{d}{c}$, or $\frac{a-b}{a} = \frac{c-d}{c}$,

that is, $a - b : a :: c - d : c$.

Obs. The foregoing results are frequently referred to in geometrical treatises in the terms annexed to them.

(7) $$ma : mb :: nc : nd$$

For, since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{ma}{mb} = \frac{nc}{nd}$,

or $ma : mb :: nc : nd.$

(8) $$ma : nb :: mc : nd$$

For, $\frac{a}{b} \times \frac{m}{n} = \frac{c}{d} \times \frac{m}{n}$, or $\frac{ma}{nb} = \frac{mc}{nd}$,

that is, $ma : nb :: mc : nd.$

(9) $$a^m : b^m :: c^m : d^m$$

For, since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a^m}{b^m} = \frac{c^m}{d^m}$,

or $a^m : b^m :: c^m : d^m.$

145. If besides the proportion $a : b :: c : d$, we have several others, as $a' : b' :: c' : d'$, $a'' : b'' :: c'' : d''$, &c. then

$$aa'a'' \ldots : bb'b'' \ldots :: cc'c'' \ldots : dd'd'' \ldots$$

For, since $\frac{a}{b} = \frac{c}{d}$, $\frac{a'}{b'} = \frac{c'}{d'}$, $\frac{a''}{b''} = \frac{c''}{d''}$, &c.

we have, by multiplying together the corresponding members of these equations,

$\frac{aa'a'' \ldots}{bb'b'' \ldots} = \frac{cc'c'' \ldots}{dd'd'' \ldots}$, or $aa'a'' \ldots : bb'b'' \ldots :: cc'c'' \ldots : dd'd'' \ldots$

146. If $a : b :: c : d$ and $c : d :: e : f$, then $a : b :: e : f$.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{c}{d} = \frac{e}{f}$, $\therefore \frac{a}{b} = \frac{e}{f}$, or $a : b :: e : f$.

147. If $a : b :: c : d$

and $b : e :: d : f$,

then $a : e :: c : f$.

For, since $\frac{a}{b}=\frac{c}{d}$, and $\frac{b}{e}=\frac{d}{f}$, $\therefore \frac{a}{b}\times\frac{b}{e}=\frac{c}{d}\times\frac{d}{f}$,

$$\text{and } \therefore \frac{a}{e}=\frac{c}{f},$$

$$\text{or } a : e :: c : f.$$

148. If $a : b :: c : d :: e : f$ &c.

then $a : b :: a + c + e \ldots : b + d + f \ldots$

For, $ad = bc$, $af = be$ (Art. 146), &c., also $ab = ba$;

$\therefore$ by addition, $a(b + d + f\ldots) = b(a + c + e\ldots)$

whence, by Art. 142, $a : b :: a + c + e \ldots : b + d + f \ldots$

149. *Geometrical Definition of Proportion.* (Euclid, book v. def. 5). The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsover of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

150. *If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.*

Let a, b, c, and d be proportional according to the algebraical definition, then $\frac{a}{b}=\frac{c}{d}$, (Art. 139),

$$\text{and } \therefore \frac{ma}{nb}=\frac{mc}{nd}, \text{ by (8) of Art. 144,}$$

o 2

therefore

$\frac{ma}{nb}$ will be $<, =,$ or > 1, according as $\frac{mc}{nd}$ is $<, =,$ or > 1:

$$\begin{aligned} \text{and } \therefore, \text{ if } ma &< nb, \\ \text{then } mc &< nd; \\ \text{or, if } ma &= nb, \\ \text{then } mc &= nd; \\ \text{or, if } ma &> nb, \\ \text{then } mc &> nd: \end{aligned}$$

and the quantities a, b, c, d are therefore proportional according to the geometrical definition.

EXAMPLES.

(1) What number must be added to a, and subtracted from b, that the sum may be to the difference as $m : n$?

Let x be the required number;

$$\text{then, } \frac{a + x}{b - x} = \frac{m}{n}$$

$$\text{whence } x = \frac{bm - an}{m + n}.$$

(2) Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

Let x be the greater, and y the less;

$$\text{then, } x : y :: x + y : 42$$
$$\text{and } x : y :: x - y : 6$$
$$\therefore \text{ (Art. 146) } x + y : 42 :: x - y : 6$$
$$\therefore \text{ alt}^{\text{do.}}, x + y : x - y :: 42 : 6 \text{ or } :: 7 : 1$$
$$\therefore \text{ comp}^{\text{do.}} \text{ \& div}^{\text{do.}}\ 2x : 2y :: 8 : 6$$
$$\text{or } x : y :: 4 : 3$$

$$\therefore\ x = \frac{4y}{3}$$

$$\text{and}\quad 4 : 3 :: \frac{y}{3} : 6$$

$$\text{whence}\quad y = 24, \text{ and } \therefore\ x = \frac{4y}{3} = 32.$$

(3) A waterman rows a given distance a, and back again in b hours; and finds that he can row c miles with the tide for d miles against it; required the rate of the tide, and the times of rowing down and up the stream, respectively.

Let x = the number of hours he rows with the tide;

then will $b - x$ = the time he rows against it;

also $\frac{a}{x}$ = rate per hour with the tide,

and $\frac{a}{b - x}$ = . . . against it

$$\therefore\ \frac{a}{x} : \frac{a}{b-x} :: c : d, \text{ by the question;}$$

$$\text{whence}\quad x = \frac{bd}{c+d}, \text{ and } \therefore\ b - x = \frac{bc}{c+d}.$$

Also $\frac{a}{x} = a \div \frac{bd}{c+d} = \frac{a(c+d)}{bd}$ = the rate down the stream = the rate of rowing + the rate of the tide; and $\frac{a}{b-x} = \frac{a(c+d)}{bc}$ = rate up the stream = the rate of rowing − the rate of the tide;

whence $\frac{a(c+d)}{bd} - \frac{a(c+d)}{bc}$ = twice the rate of the tide;

and $\therefore$ the rate of the tide $= \frac{a(c^2 - d^2)}{2bcd}$.

EXAMPLES FOR PRACTICE.

(1) Divide the number 60 into two parts, such that their product shall be to their sum as 40 to 3.

Ans. 40 and 20.

(2) What number must be subtracted from a and b that the differences may have the ratio of $m : n$?

Ans. $\frac{an - bm}{n - m}$.

(3) What two numbers are those of which the difference, sum and product are as the numbers 2, 3 and 5, respectively?

Ans. 10 and 2.

(4) A packet, sailing from Dover with a fair wind, arrives at Calais in two hours; and on its return, the wind being contrary, it sails six miles an hour slower than it went. Now, when it is half-way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done, had the wind not changed, in the proportion of 6 : 7. Required the rates of sailing, and the distance between Dover and Calais.

Ans. The distance is 22 miles; and the rates of sailing are 11, 5 and 7 miles an hour.

VARIATION.

151. When two quantities are so mutually dependent, that any change in the one is attended with a proportional change in the other, they are said to vary, the one as the other.

152. One quantity is said to *vary directly* as another, when if one be increased or diminished the other is increased or diminished in the same proportion.

Thus, if a man travel a given time, the distance he travels is increased as his speed is increased.

Let S and D respectively represent the original speed and distance, and s and d the altered speed and distance;

$$\text{then } S : s :: D : d.$$

153. One quantity is said to *vary inversely* as another, when if one be increased the other is diminished, and if one be diminished the other is increased, always in the same proportion.

Thus, if a man travel a given distance, the time he is travelling is diminished as his speed is increased, and *vice versâ.*

Let S represent the speed as before, and T the original time, and s and t the altered speed and time respectively;

$$\text{then} \quad S : s :: \frac{1}{T} : \frac{1}{t};$$

whence it appears that a quantity which varies inversely as another, varies as the reciprocal of the other.

From this proportion we obtain

$$S : s :: t : T.$$

154. A quantity is said to *vary as others jointly,* when it varies as their product.

Thus, if D varies as TS, it is said to vary as T and S jointly, and

$$D : d :: TS : ts.$$

155. Variation is expressed by the sign $\propto$ placed between the quantities so related; thus

$A \propto B$ signifies that A varies as B directly,

$A \propto \frac{1}{B}$ signifies that A varies inversely as B,

and $A \propto BC$ signifies that A varies jointly as B and C.

Also, $A \propto \frac{B}{C}$ signifies that A varies directly as B and inversely as C; in which case, if A be changed to a, whilst B is changed to b and C to c, then will

$$A : a :: \frac{B}{C} : \frac{b}{c}.$$

156. ***When the increase or decrease of one quantity depends upon the increase or decrease of two others, and it appears that when either of these two quantities is constant, or invariable, the first varies as the other, then, when they both vary, the first varies as their product.***

Thus, if V be the velocity of a body moving uniformly, T the time of motion, and S the space described; if T be constant $S \propto V$, if V be constant $S \propto T$, but if neither be constant, then $S \propto TV$.

Let s, v, t be any other velocity, space, and time; and s' be the space described with the velocity v in the time T; then

$$S : s' :: V : v, \text{ (T being the same in both cases,)}$$

$$s' : s :: T : t, \text{ (v being the same in both;)}$$

$$\therefore\ S : s :: TV : tv,$$

$$\text{and } \therefore\ S \propto TV.$$

ARITHMETICAL PROGRESSION.

157. Quantities are said to be in *Arithmetical Progression*, when they increase or decrease by a *common difference*.

Thus 1, 4, 7, 10, &c.; a, $a + d$, $a + 2d$, $a + 3d$, &c.; a, $a - d$, $a - 2d$, $a - 3d$, &c. are in arithmetical progression.

158. In the general series a, $a + d$, $a + 2d$, &c. the n^{th} or general term will evidently be $a + (n - 1)d$, since the coefficient of d in every term is one less than the number which denotes the place of the term in the series.

159. *To find the sum of a series of quantities in arithmetical progression.*

Let a be the first term, d the common difference, n the number of terms, and s the sum of the series; then,

$$\begin{aligned} s &= a + (a + d) + (a + 2d) + \ldots. \\ &\qquad + \{a + (n - 2)d\} + \{a + (n - 1)d\} \\ &= \{a + (n - 1)d\} + \{a + (n - 2)d\} + \ldots. \\ &\qquad + (a + 2d) + (a + d) + a \end{aligned}$$

and, by adding together the corresponding terms, we have

$$\begin{aligned} 2s &= \{2a + (n - 1)d\} + \{2a + (n - 1)d\} + \text{\&c. to } n \text{ terms} \\ &= n\{2a + (n - 1)d\} \end{aligned}$$

and $\therefore s = \dfrac{n}{2}\{2a + (n - 1)d\}$

$= \dfrac{n}{2}(a + l)$, if we put l for the last term.

Hence, the sum of a series of quantities in arithmetical progression is found by multiplying the sum of the first and last terms by half the number of terms.

160. By means of the equations $l = a + (n - 1)d$ and $s = \dfrac{n}{2}\{2a + (n - 1)d\}$, any three of the quantities s, a, n, d, l being given, the two others may be found.

If d be negative, or the series a decreasing one, these equations will become

$$l = a - (n - 1)d \text{ and } s = \frac{n}{2}\{2a - (n - 1)d\}.$$

EXAMPLES.

(1) Find the n^{th} term and the sum of n terms of the series of odd numbers, 1, 3, 5, 7, &c.

Generally, $l = a + (n-1)\,d$, and $s = \frac{n}{2}\,\{2a + (n-1)\,d\}$;

here $a = 1$, and $d = 2$; whence, by substitution,

$$l = 1 + (n-1)\,2 = 2n - 1,$$

$$\text{and } s = \frac{n}{2}\,\{2 + (n-1)\,2\} = n^2.$$

(2) Find the n^{th} term and the sum of n terms of the series of even numbers, 2, 4, 6, 8, &c.

Here $a = 2$ and $d = 2$,

$$\therefore\ l = 2 + (n-1)\,2 = 2n,$$

$$\text{and } s = \frac{n}{2}\,\{4 + (n-1)\,2\} = n\,(n+1).$$

(3) Required the 7^{th} term, and the sum of 7 terms of the series 11, 8, 5, &c.

Here $a = 11$, $n = 7$, and $d = -3$

$$\therefore\ l = -7, \text{ and } s = 14.$$

(4) If the first term of an arithmetical progression be 2, and the sum of 8 terms 100, what is the common difference?

$$\text{Since } s = \frac{n}{2}\,\{2a + (n-1)\,d\}$$

$$2a + (n-1)\,d = \frac{2s}{n}$$

$$\text{and } d = \frac{2s}{n\,(n-1)} - \frac{2a}{n-1} = \frac{2s - 2an}{n\,(n-1)}.$$

In the case proposed, $s = 100$, $a = 2$, and $n = 8$;

$$\therefore\ d = \frac{200 - 32}{56} = 3.$$

Hence, the series is 2, 5, 8, 11, 15, 18, 21, 24.

(5) If the sum of 9 terms of an arithmetical progression be 171, and the common difference 4, what is the first term?

$$\text{Since } s = \frac{n}{2}\{2a + (n-1)\,d\}$$

$$a = \frac{s}{n} - \frac{(n-1)\,d}{2};$$

and substituting 171 for s, 9 for n, and 4 for d,

$$a = \frac{171}{9} - \frac{32}{2} = 3.$$

Whence, the series is 3, 7, 11, 15, 19, 23, 27, 31, 35, 39.

EXAMPLES FOR PRACTICE.

(1) Find the sum of 12 terms of the series 8, 15, 22, &c.
Ans. 558.

(2) Required the 7th term, and the sum of 7 terms of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, &c. *Ans.* $-\frac{1}{2}$, and 0.

(3) Required the number of terms and the common difference, when the sum is 2.748, the first term .034, and the last term .0576. *Ans.* 60, and .0004.

(4) What number of terms of the series 54, 51, 48, &c. must be taken to make their sum 513? *Ans.* 18 or 19.

(5) The number of deaths in a besieged garrison amounted to 6 daily; and, allowing for this diminution, their stock of provisions was sufficient to last 8 days. But, on the evening of the 6th day, 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the

stock of provisions unconsumed at the end of the 6th day sufficient to support 6 men for 61 days; it is required to find how long it would support the garrison, and the number of men alive when the provisions were exhausted.

Ans. 6 days, and 26 men.

161. *To insert any number of arithmetical means between any two given quantities.*

Let a and l be the two given quantities, and m the number of means required to be inserted between them; then, since the number of terms $= m + 2$, we shall have

$$l = a + (m + 1)\,d, \text{ and } \therefore d = \frac{l - a}{m + 1}:$$

whence the series will be

$$a,\ a + \frac{l - a}{m + 1},\ a + 2\frac{l - a}{m + 1},\ \&\text{c. } a + m\frac{l - a}{m + 1},\ l;$$

$$\text{or} \quad a,\ \frac{ma + l}{m + 1},\ \frac{(m - 1)\,a + 2l}{m + 1},\ \&\text{c. } \frac{a + ml}{m + 1},\ l.$$

Ex. Insert three arithmetic means between 117 and 477.

$$l = a + (m + 1)\,d$$

$$\text{or} \quad 477 = 117 + 4d, \text{ and } \therefore d = 90;$$

whence the series is 117, 207, 297, 387, 477.

The same might be obtained by substituting for a, m, and l, their values, in the general series.

EXAMPLES FOR PRACTICE.

(1) Insert 4 arithmetic means between 193 and 443.

Ans. The means are 243, 293, 343, 393.

(2) Find 2 arithmetic means between -3 and 3.

Ans. -1 and 1.

(3) Insert 9 arithmetic means between 1 and -1.

Ans. $\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, -\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}$.

162. ***Any two terms and their places being given, to determine the series.***

Let P and Q represent the p^{th} and q^{th} terms respectively, and the rest as before; then,

$$P = a + (p-1)\,d, \quad \text{and} \quad Q = a + (q-1)\,d;$$

$$\text{whence} \quad P - Q = (p-q)\,d, \quad \text{and} \;\therefore\; d = \frac{P-Q}{p-q};$$

$$\therefore\; a = P - (p-1)\,d = P - \frac{p-1}{p-q}(P-Q)$$

$$= \frac{Q(p-1) - P(q-1)}{p-q}.$$

Hence, the first term and common difference being found, the series may be determined.

$$\text{Moreover, the } n^{\text{th}} \text{ term} = a + (n-1)\,d$$

$$= \frac{Q(p-n) - P(q-n)}{p-q};$$

$$\text{and the sum of } n \text{ terms} = \frac{n}{2}\{2a + (n-1)\,d\}$$

$$= \frac{n}{2}\left\{\frac{Q(2p-n-1) - P(2q-n-1)}{p-q}\right\}.$$

GEOMETRICAL PROGRESSION.

163. *A Geometrical Progression* is a series of quantities in continued proportion, which therefore increase or decrease by a *common ratio.*

If a be the first term and r the common ratio, the series will be a, ar, ar^2, ar^3, &c.

$$\text{For} \quad r = \frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \&c.$$

$$\text{or} \quad a : ar :: ar : ar^2 :: ar^2 : ar^3 \ \&c.$$

164. Hence, if the series be given, the common ratio may be found by dividing the second term by the first, or any other term by that which immediately precedes it.

165. *If quantities be in geometrical progression, their differences are in geometrical progression.*

Let a, ar, ar^2, ar^3, &c. be the quantities: their differences $ar - a$, $ar^2 - ar$, $ar^3 - ar^2$, &c. form a geometrical progression, whose first term is $ar - a$, and common ratio r.

166. *Quantities in geometrical progression are proportional to their differences.*

For, a, ar, ar^2, ar^3, &c. being the quantities,

$$a : ar :: ar - a : ar^2 - ar :: ar^2 - ar : ar^3 - ar^2 \ \&c.$$

167. *In any geometrical progression, the first term is to the third, as the square of the first to the square of the second.*

Let a, ar, ar^2, ar^3, &c. be the progression; then

$$a : ar^2 :: a^2 : a^2r^2.$$

In the same manner it may be shewn, that the first term is to the $(n+1)^{\text{th}}$, as the first raised to the n^{th} power, to the second raised to the same power.

168. *The terms taken at equal intervals, in a geometrical progression, are in geometrical progression.*

For, of the progression a, ar, ar^2, ... ar^n, ... ar^{2n}, ... &c. the terms a, ar^n, ar^{2n}, &c., taken at the interval of n terms, form a geometrical progression, whose common ratio is r^n.

169. *To find the sum of a series of quantities in geometrical progression.*

Let S represent the sum, a the first term, r the common ratio, l the last term, and n the number of terms; then $l = ar^{n-1}$, and

$$S = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1};$$

$$\therefore\ rS = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n,$$

and, subtracting the upper line from the lower,

$$(r-1)\,S = -a + ar^n$$

$$\text{and } \therefore\ S = a\,\frac{r^n - 1}{r-1}, \text{ or } \frac{rl - a}{r-1}.$$

EXAMPLES.

(1) Required the 7th term, and sum of 7 terms of the series 1, 2, 4, &c.

Here $a=1$, $r=2$, $n=7$; whence, putting l for the 7th term,

$$l = ar^{n-1} = 2^6 = 64;$$

$$\text{and}\quad S = a\,\frac{r^n - 1}{r-1} = 2^7 - 1 = 127.$$

(2) Find the nth term and the sum of n terms of the series $c^2 - x^2$, $c + x$, $\dfrac{c+x}{c-x}$, &c.

In this case

$$a = c^2 - x^2, \text{ and } r = \frac{c+x}{c^2 - x^2} = \frac{1}{c-x} \text{ (Art. 164)};$$

$$\therefore\ l = ar^{n-1} = (c^2 - x^2)\,\frac{1}{(c-x)^{n-1}}$$

$$= \frac{(c+x)(c-x)}{(c-x)^{n-1}} = \frac{c+x}{(c-x)^{n-2}};$$

and $$S = a\,\frac{r^n - 1}{r - 1} = (c^2 - x^2)\,\frac{\dfrac{1}{(c-x)^n} - 1}{\dfrac{1}{c-x} - 1} =$$

$$\frac{c+x}{(c-x)^{n-2}} \cdot \frac{1-(c-x)^n}{1-(c-x)}, \text{ or } \frac{c+x}{(c-x)^{n-2}} \cdot \frac{(c-x)^n - 1}{c-x-1}.$$

(3) Given S, r, n, to find a.

$$S = a\,\frac{r^n - 1}{r - 1}; \quad \text{whence } a = \frac{r-1}{r^n - 1}\,S.$$

(4) Given S, r, l, to find a.

$$S = \frac{rl - a}{r - 1}; \quad \text{whence } a = rl - (r-1)\,S.$$

(5) Given S, a, r, to find n.

$$S = a\,\frac{r^n - 1}{r - 1}; \quad \text{whence } r^n = \frac{s}{a}\,(r-1) + 1.$$

From this equation n may be obtained, but its solution generally requires the aid of *Logarithms.*

EXAMPLES FOR PRACTICE.

(1) Required the 12th term, and the sum of 12 terms of the series 1, 3, 9, 27, &c.

Ans. 177147, and 265720.

(2) Find the sum of 10 terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, &c.

Ans. $2\frac{18659}{19683}$.

(3) Required the sum of 6 terms of the series 1, $-\frac{3}{4}$, $\frac{9}{16}$, $-\frac{27}{64}$, &c.

Ans. $\frac{481}{1024}$.

(4) Given a, n, l, to find r. *Ans.* $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$.

(5) Given a, n, l, to find S. *Ans.* $S = \dfrac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$.

(6) A population, which increased every year in the same proportion, in four years became from 10000 to 14641. By what part did it increase yearly? *Ans.* By the 10th part.

(7) What is the sum of n terms of the geometrical progression a, b, $\dfrac{b^2}{a}$, ... $\dfrac{b^{n-1}}{a^{n-2}}$? *Ans.* $\dfrac{a^n - b^n}{(a-b)\,a^{n-2}}$.

170. *To find the sum of a decreasing Geometrical series, continued in infinitum.*

When r is a proper fraction, as n increases, the value of r^n, or of ar^n, decreases, and when n increases without limit, ar^n becomes less, with respect to a, than any magnitude that can be assigned; whence

$$S = \frac{-a}{r-1}, \text{ or } \frac{a}{1-r}.$$

This quantity, which we call the sum of the series, is the limit to which the sum of the terms approaches, but never actually attains: it is however the true representative of the series continued *sine fine;* for the division of a by $1-r$ will reproduce this series.

If r be > 1, the terms continually *increase*, and therefore the sum of an infinite series of such terms is infinite.

EXAMPLES.

(1) Required the sum of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c. *in inf.*

Here $a = 1$, and $r = \frac{1}{3}$;

$$\therefore\ S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = 1\tfrac{1}{2}.$$

(2) Sum the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} +$ &c. *in infinitum.*

$$\text{Here} \quad a = 1, \text{ and } r = -\tfrac{1}{2};$$

$$\therefore \; S = \frac{a}{1-r} = \frac{1}{1+\frac{1}{2}} = \tfrac{2}{3}.$$

(3) The sum of a geometrical series is 27, and the common ratio $\frac{2}{3}$; what is the first term?

$$\text{Here} \quad S = 27, \text{ and } r = \tfrac{2}{3};$$

therefore by substituting these values in the general formula $S = \dfrac{a}{1-r}$, we have

$$27 = \frac{a}{1-\frac{2}{3}};$$

$$\text{whence} \quad a = 27 - 18 = 9.$$

EXAMPLES FOR PRACTICE.

(1) Sum the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c. *in infinitum.*

Ans. $S = 1$.

(2) Sum the series 40 $17\frac{1}{7} + 7\frac{17}{49} -$ &c. *in infinitum.*

Ans. 28.

(3) The sum of n terms of a geometrical series is $\dfrac{1}{\sqrt{x^n}} \cdot \dfrac{\sqrt{x^n} \pm \sqrt{a^n}}{\sqrt{x} + \sqrt{a}}$, and the common ratio $-\sqrt{\dfrac{a}{x}}$; find the series. *Ans.* $\dfrac{1}{\sqrt{x}} - \dfrac{\sqrt{a}}{x} + \dfrac{a}{x\sqrt{x}} - \dfrac{a\sqrt{a}}{x^2} +$ &c.

171. *To insert any number of geometrical means between any two given quantities.*

Let a and l be the two given quantities, m the number of means required to be inserted between them; then, since the number of terms $= m + 2$,

$$l = ar^{n-1} = ar^{m+1}; \quad \text{and} \therefore r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}}:$$

and r being thus found, the means may be determined from the series a, ar, ar^2, ar^3, &c.

Ex. Insert three geometric means between 39 and 3159.

$$\text{Here} \quad a = 39, \quad l = 3159, \quad \text{and } m = 3;$$

$$\therefore 3159 = 39r^4$$

$$\therefore r = \left(\frac{3159}{39}\right)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = 3:$$

whence the series is 39, 117, 351, 1053, 3159.

EXAMPLES FOR PRACTICE.

(1) Find three geometric means between 9 and $\frac{1}{9}$.

Ans. 3, 1, $\frac{1}{3}$.

(2) The first and sixth terms of a geometrical series are 1 and 2; required the series.

Ans. 1, $2^{\frac{1}{5}}$, $2^{\frac{2}{5}}$, $2^{\frac{3}{5}}$, $2^{\frac{4}{5}}$, 2, &c.

172. *Any two terms of a geometrical progression and their places being given, to find the series.*

Let the p^{th} and q^{th} terms be P and Q, respectively, and the rest as before; then $P = ar^{p-1}$, and $Q = ar^{q-1}$;

$$\text{whence } \frac{P}{Q} = r^{p-q}, \quad \text{and} \therefore r = \left(\frac{P}{Q}\right)^{\frac{1}{p-q}}:$$

$$\text{also} \quad a = \frac{P}{r^{p-1}} = P\left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}} = \left(\frac{Q^{p-1}}{P^{q-1}}\right)^{\frac{1}{p-q}}:$$

and, the first term and the common ratio being determined, the series becomes known.

The n^{th} term $= \left(\dfrac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}}$; and the sum of n terms $=$

$$\left(\frac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}}\left\{\frac{P^{\frac{n}{p-q}}-Q^{\frac{n}{p-q}}}{P^{\frac{1}{p-q}}-Q^{\frac{1}{p-q}}}\right\}=\left(\frac{P^{q-n}}{Q^{p-n}}\right)^{\frac{1}{q-p}}\left\{\frac{Q^{\frac{n}{q-p}}-P^{\frac{n}{q-p}}}{Q^{\frac{1}{q-p}}-P^{\frac{1}{q-p}}}\right\}.$$

173. The successive periods of *Recurring Decimals*, taken separately, form geometrical progressions, having, respectively, either $\frac{1}{10}$, or some power of it for their common ratios. Thus,

$$.333\ \&c. = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \&c.\ \textit{in infinitum}.$$

$$.231231\ \&c. = \frac{231}{10^3} + \frac{231}{10^6} + \frac{231}{10^9} + \&c.\ \textit{in infinitum}.$$

Their sums may therefore be expressed by vulgar fractions, by means of the general formula of Art. 170; but in practice the most convenient methods are those explained in the *Arithmetic*, putting S for the value of the required fraction. The generality of these methods may be shewn as follows.

(1) To transform $.PPP...$ *in infinitum*, where P contains p digits, to its equivalent vulgar fraction.

$$\text{If} \quad S = .PPP...$$
$$\text{then will} \quad 10^p S = P.PPP...$$
$$\therefore \text{ by subtraction,} \quad (10^p - 1)\,S = P$$
$$\text{whence} \quad S = \frac{P}{10^p - 1}.$$

(2) To transform $.PQQQ...$ *in infinitum*, where P and Q contain p and q digits respectively, into its equivalent vulgar fraction.

$$\text{Let } S = .PQQQ\ldots$$

$$\therefore\ 10^{p+q}S = PQ.QQQ\ldots$$

$$\text{and } 10^{p}S = P.QQQ\ldots$$

$$\therefore\ (10^{p+q} - 10^{p})\,S = PQ - P$$

$$\text{whence } S = \frac{PQ-P}{10^{p+q}-10^{p}} = \frac{PQ-P}{10^{p}(10^{q}-1)}.$$

We shall here introduce the proofs of the rules for the *Multiplication* and *Division of Decimals.*

174. *To prove the rule for the Multiplication of Decimals.*

Let $.M$ be the multiplier and $.N$ the multiplicand, and let them contain m and n decimal places respectively; then

$$.M = \frac{M}{10^{m}}, \text{ and } .N = \frac{N}{10^{n}},$$

$$\text{whence } .M \times .N = \frac{M}{10^{m}} \times \frac{N}{10^{n}} = \frac{MN}{10^{m+n}};$$

from which we deduce the general conclusion, that the multiplication of decimals is performed as in whole numbers; and that the product contains as many decimal places as the multiplier and multiplicand together.

175. *To prove the rule for the Division of Decimals.*

Let $.M$ be the dividend and $.N$ the divisor, and let them contain m and n decimal places respectively; then

$$\text{since } .M = \frac{M}{10^{m}} \text{ and } .N = \frac{N}{10^{n}},$$

$$\frac{.M}{.N} = \frac{M}{10^{m}} \div \frac{N}{10^{n}} = \frac{M}{10^{m}} \times \frac{10^{n}}{N} = \frac{M}{N} \times \frac{1}{10^{m-n}} \text{ or } \frac{M}{N} \times 10^{n-m},$$

according as m is greater or less than n.

First, let m be greater than n; then it is evident, that after the division is effected as in integers, the quotient must contain $m - n$ decimal places. Next, let m be less than n, then it appears that we must affix to the quotient $n - m$ ciphers, and that the result will be an integer.

HARMONICAL PROGRESSION.

176. An *Harmonical Progression* is a series of quantities of which, if any three consecutive terms be taken, the first is to the third as the difference between the first and second is to the difference between the second and third.

Thus, if a_1, a_2, a_3, a_4, &c. be the consecutive terms of an harmonical progression,

$$a_1 : a_3 :: a_1 - a_2 : a_2 - a_3;$$

$$a_2 : a_4 :: a_2 - a_3 : a_3 - a_4; \text{ \&c.}$$

177. *The reciprocals of quantities in harmonical progression are in arithmetical progression.*

For, let a_1, a_2, a_3, a_4, &c. be the quantities; then

$$a_1 : a_3 :: a_1 - a_2 : a_2 - a_3,$$

$$\therefore a_1a_2 - a_1a_3 = a_1a_3 - a_2a_3$$

and dividing by $a_1a_2a_3$,

$$\frac{1}{a_3} - \frac{1}{a_2} = \frac{1}{a_2} - \frac{1}{a_1} = \delta, \text{ suppose};$$

$$\text{similarly, } \frac{1}{a_4} - \frac{1}{a_3} = \frac{1}{a_3} - \frac{1}{a_2} = \delta, \text{ \&c.}$$

The quantities $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}$, &c. therefore form an arithmetical progression, whose common difference is δ.

178. Hence, if a_n be the n^{th} term of an harmonical progression, $\frac{1}{a_n}$ will be the n^{th} term of the corresponding arithmetical progression: and any number of harmonical means between a_1 and a_n will be the reciprocals of the same number of arithmetical means between $\frac{1}{a_1}$ and $\frac{1}{a_n}$.

VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

179. The different orders in which any given number of quantities can be arranged, when only part of them are taken together, are called their *Variations*.

Thus, the variations of *a*, *b* and *c*, taken two and two together, are *ab*, *ac*, *ba*, *bc*, *ca* and *cb*.

180. The different orders in which any number of quantities can be arranged, when the whole are taken together, are called their *Permutations*.

Thus the permutations of *a* and *b* are *ab* and *ba*; of *a*, *b* and *c* are *abc*, *acb*, *bac*, *bca*, *cab* and *cba*.

Without attending to the above distinction, the terms *Variations*, *Permutations*, *Changes*, &c. are frequently used promiscuously.

181. The different collections that can be made of any given number of quantities, without regard to the order in which they are placed, are called their *Combinations*.

Thus, *ab*, *ac* and *bc* are the combinations of the three letters *a*, *b*, *c*, taken two and two; *ab* and *ba*, though different variations, forming the same combination.

182. The number of variations of n things, taken r and r together, is denoted thus, ${}^{n}V_r$; the number of permutations of n things thus, ${}^{n}P$; and the number of combinations, when they are taken r and r together, thus, ${}^{n}C_r$.

183. *To find the number of variations of* n *different things, taken* r *and* r *together.*

Let $a_1, a_2, a_3, \dots a_n$ represent the n things whose variations are required.

Then the number of variations, when the quantities are taken separately, is clearly n.

Also, the number, when they are taken two and two together, is $n(n-1)$; for a_1 may be placed before each of the rest, and thus form $n-1$ variations, two and two, in which a_1 stands first: and the same may be done with $a_2, a_3, a_4, \dots a_n$, when there will be $n-1$ variations in which each stands first; and therefore on the whole there will be $n(n-1)$ such variations; or ${}^{n}V_2 = n(n-1)$.

Again, the number of variations, when they are taken three and three, is $n(n-1)(n-2)$; for, since it appears that there are $n(n-1)$ variations of n things taken two and two, there are $(n-1)(n-2)$ variations of the $(n-1)$ things $a_2, a_3, a_4, \dots a_n$, taken two and two together; and by prefixing a_1 to each of these, there will be $(n-1)(n-2)$ variations of n things, taken three and three, of which a_1 stands first: and since there are the same number of variations in which $a_2, a_3, \dots a_n$ respectively occupy the first place, there are on the whole $n(n-1)(n-2)$ variations of n things taken three and three; or ${}^{n}V_3 = n(n-1)(n-2)$.

By a similar process of reasoning, we should obtain for the number of variations of n things taken four and four together, $n(n-1)(n-2)(n-3)$; where there are four

factors which are the natural numbers descending from n: and the law which is thus found to hold for the number of variations of n things taken two and two, three and three, four and four together, may be extended by induction to the expression for the number of variations, when any required number (r) are taken together.

In order however to demonstrate the correctness of this induction, we must shew that, if this law be true for any one class of variations, it is true for the next class.

Assuming, therefore, that

$$^nV_r = n(n-1)(n-2)\ldots(n-\overline{r-1}); \qquad (a)$$

it is required to prove that

$$^nV_{r+1} = n(n-1)(n-2)\ldots(n-r).$$

Omitting a_1, and forming all the variations of the $n-1$ remaining quantities, a_2, a_3, ... a_n, taken r and r together, the expression for their number will be found by putting $n-1$ in the place of n in the formula (a), or, which is the same thing, by subtracting 1 from each of its factors, when it becomes

$$(n-1)(n-2)(n-3)\ldots(n-r):$$

before each of these variations a_1 may be placed, and there will therefore be $(n-1)(n-2)(n-3)\ldots(n-r)$ variations in which a_1 occupies the first place: there will clearly be the same number of such variations in which each of the remaining letters stands first; and therefore the whole number of variations is n times that number; or

$$^nV_{r+1} = n(n-1)(n-2)\ldots(n-r).$$

The law of the formation of the expression for the number of variations, if true for one class of variations, is therefore

true for the next class: and it has been shewn to be true when n things are taken two and two, and three and three together, it is therefore, by successive inferences, true when any required number (r) of them are taken together.

Ex. Let there be five things; then, by the above formula, we shall have

$$^5V_1 = 5, \quad ^5V_2 = 5.4 = 20, \quad ^5V_3 = 5.4.3 = 60,$$
$$^5V_4 = 5.4.3.2 = 120.$$

184. If $r = n$, or the variations become permutations, we have

$$^nP = n(n-1)(n-2) \dots 3.2.1, \text{ or } 1.2.3 \dots n.$$

Ex. How many changes may be rung on eight bells? Here $^8P = 8.7.6.5.4.3.2.1 = 40320$, the number required.

185. *To find an expression for the number of permutations of* n *things, when* p *of them are of one kind,* q *of another,* r *of another,* &c.

The expression for the number of permutations, supposing them all different from each other, is $1.2 \dots n$: if p of these quantities become identical, the permutations which arise from their interchange with each other, $1.2 \dots p$ in number, for any assigned position of the other quantities, are reduced to one: the number of permutations, therefore, when all of them are different, is $1.2 \dots p$ times as great as when p of them become identical: whence

$$\frac{1.2 \dots n}{1.2 \dots p}$$

is the expression for the number of permutations when p of them are alike.

If, in addition to p quantities which become identical, there are q others, which, though different from the former, are identical with each other, then there are $1.2 \ldots q$ permutations corresponding to their interchange with each other, which are reduced to one, for any given position of the other quantities: the expression for the number of permutations under these circumstances, therefore, becomes

$$\frac{1.2 \ldots n}{1.2 \ldots p.1.2 \ldots q}.$$

In the same manner it may be shewn that if, besides these, there are r other identical quantities, the expression becomes

$$\frac{1.2 \ldots n}{1.2 \ldots p.1.2 \ldots q.1.2 \ldots r}:$$

and so on for any number of identical quantities.

EXAMPLES.

(1) To find the number of permutations (P) of the letters in the word *Algebra*.

In this case $n = 7$, and the a appears twice;

$$\therefore \; P = \frac{7.6.5.4.3.2.1}{1.2} = 2520.$$

(2) To find the number of permutations of the letters in the product $a^2b^3c^4$ written at full length.

Here $n = 9$, and the letter a appears twice, b three times, and c four times;

$$\therefore \; P = \frac{9.8.7.6.5.4.3.2.1}{1.2.1.2.3.1.2.3.4} = 12600.$$

EXAMPLES FOR PRACTICE.

(1) Find the number of permutations of the letters in the word *Analysis*. *Ans.* 10080.

(2) Find the number of permutations of the product ab^2c^3. *Ans.* 60.

(3) What is the number of permutations of a and b in the expression $a^{n-r}b^r$? *Ans.* $\dfrac{n(n-1)\dots(n-\overline{r-1})}{1.2.3\dots r}$.

The answer to this question, it may be observed, will be found to correspond with the expression for the number of combinations of n things, taken r and r together.

186. *To find the number of combinations of* n *things taken* r *and* r *together.*

$$\text{Now, } {}^nV_r = n(n-1)(n-2)\dots(n-\overline{r-1});$$

and each of the combinations in ${}^nC_r$ admits of as many permutations as the r quantities in one of them will form, that is, of $r(r-1)\dots 3.2.1$ permutations; hence,

$$\begin{aligned} &{}^nC_r \times r(r-1)\dots 3.2.1 \\ &\qquad = {}^nV_r = n(n-1)(n-2)\dots(n-\overline{r-1}); \end{aligned}$$

$$\text{and } \therefore\ {}^nC_r = \frac{n(n-1)(n-2)\dots(n-\overline{r-1})}{1.2.3\dots r}.$$

Ex. 1. How many ternary combinations, that is, when taken three and three, can be made with the twenty-six letters of the Alphabet?

$$\text{Here } n = 26, \text{ and } r = 3;$$

$$\therefore\ {}^nC_r = {}^{26}C_3 = \frac{26\times 25\times 24}{1\times 2\times 3} = 2600.$$

Ex. 2. Of 100 soldiers, in how many ways can 4 be taken for sentinels? *Ans.* 3921225.

187. Since $^nC_r = \dfrac{n(n-1)\dots(n-\overline{r-2})(n-\overline{r-1})}{1.2\dots(r-1)r}$, we have, by writing $r-1$ for r,

$$^nC_{r-1} = \frac{n(n-1)\dots(n-\overline{r-3})(n-\overline{r-2})}{1.2\dots(r-2)(r-1)}$$

$$\text{and } \therefore\ ^nC_r = {}^nC_{r-1} \times \frac{n-(r-1)}{r}:$$

whence it appears that, in forming these combinations, we may form the second and each succeeding combination by multiplying that which immediately precedes it by the fraction $\dfrac{n-(r-1)}{r}$.

Ex. Let $n=6$, and $r=1, 2, 3, 4, 5, 6$, successively; then

$$^6C_1=6,\quad ^6C_2=6\times\tfrac{5}{2}=15,\quad ^6C_3=15\times\tfrac{4}{3}=20,$$

$$^6C_4=20\times\tfrac{3}{4}=15,\quad ^6C_5=15\times\tfrac{2}{5}=6,\quad ^6C_6=6\times\tfrac{1}{6}=1.$$

188. If of n quantities $a_1, a_2, a_3, \dots a_n$, we form a single combination by taking r of them, the remaining $n-r$ quantities will form a *supplementary* combination: and this will be the case for every combination formed by taking r out of n quantities: hence, there will be as many supplementary as primary combinations, or $^nC_r = {}^nC_{n-r}$.

This may also be seen by putting $n-r$ for r in the expression for ^nC_r.

189. *To find the value of* r *that* ^nC_r *may be the greatest possible.*

If in the expression for ^nC_r, which is $\dfrac{n(n-1)\dots(n-\overline{r-1})}{1.2.3\dots r}$, we make $r=1, 2, 3,$ &c. successively, it will increase

until the additional term introduced into the numerator becomes equal to or less than the corresponding additional term in the denominator: now, the r^{th} term of the numerator is $n-(r-1)$, and the corresponding term in the denominator is r; if these be equal to each other, or $n-r+1=r$, we have $r=\frac{n+1}{2}$; and since r is a whole number, n must be an odd number: there is, therefore, in this case, the same number, and also the greatest number, of combinations, when $r=\frac{n+1}{2}$ and when it $=\frac{n-1}{2}$ such combinations being supplementary to each other.

If n be an even number, the number of combinations is greatest when $r=\frac{n}{2}$: for, in this case, $n-r+1$ becomes $\frac{n}{2}+1$, which is greater than the corresponding value of r, or $\frac{n}{2}$; whilst the next succeeding value of $n-r+1$, which is $\frac{n}{2}$, is less than the corresponding value of r, or of the last term of the denominator, which is $\frac{n}{2}+1$.

Ex. 1. Of nine things, how many must be taken together, that the number of combinations may be the greatest possible?

If r be the required number, since the given number of things (n) is odd, we may have $r=\frac{n-1}{2}$ or $\frac{n+1}{2}=4$ or 5; the number of combinations in either case being 126, which is greater than when any other numbers are taken together.

Ex. 2. Of six things, how many must be taken together, that the number of combinations may be the greatest possible? *Ans.* 3.

LOGARITHMS.

190. The value of x which satisfies the equation $a^x = n$ is called the *Logarithm* of n to the *base* or *radix* a.

Thus, if we suppose $a = 10$ and $n = 100$, 1000, &c. successively, then, since $100 = 10^2$, $1000 = 10^3$, &c., we shall have $\log 100 = 2$, $\log 1000 = 3$, &c., and consequently the logarithm of any number intermediate to 100 and 1000, or to 1000 and 10000, &c., some mixed number between 1 and 2, 2 and 3, &c.; as $\log 15 = 1.760913$, $(10)^{1.760913}$ being nearly equal to 15.

191. The indices of the same base, corresponding to the successive values of n, constitute what is called a *system* of logarithms.

Thus, if the base be 10, which is the base of our scale of numeration; the indices form the *tabular system*, being those which are registered in the ordinary tables of logarithms. It is also called *Briggs' system*. This system is used exclusively in arithmetical calculations.

If the base be 2.7182818..., which is the sum of the series

$$1 + 1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \ldots \textit{ in infinitum},$$

the corresponding indices form the *Napierian system*, the base being selected by Lord Napier, the inventor of logarithms. This system is almost exclusively used in algebraical formulæ, and the base is briefly represented by the letter e.

192. Since $a^0 = 1$, whatever be the value of a, (Art. 51), $\log 1 = 0$ in any system.

193. The quantities e and 10 are the only bases of logarithms which are used in analytical enquiries, although the

symbol a is employed in general algebraical investigations to denote any base whatever, whether different from e and 10, or the same.

194. Systems of logarithms are variously distinguished by different writers.

Thus, for the common, Briggs', or tabular logarithms, we find Com. log n, Brigg. log n, t log n, &c.

For the Napierian, which are also called *Natural* and *Hyperbolic* logarithms,

Nap. log n, Nat. log. n, Hyp. log n, &c.

But the following notation is now frequently used, when the distinction is required:

$$\log_{10} n, \quad \log_e n, \quad \log_a n,$$

according as the base is 10, e, or a.

195. The logarithm of any quantity being such an index as makes the corresponding power of the assumed base equal to that quantity, the properties of logarithms are the properties of the indices of the same base. Hence,

(1) *In any system, the logarithm of a product, consisting of any number of factors, is equal to the sum of the logarithms of those factors.*

For, if a be the base of the system, and x, x', x'', &c., the logarithms of n, n', n'', &c., we have, by definition,

$$n = a^x, \quad n' = a^{x'}, \quad n'' = a^{x''}, \text{ \&c.}$$

$$\text{whence } nn'n'' \ldots = a^x a^{x'} a^{x''} \ldots = a^{x+x'+x''} \ldots$$

$$\text{and } \therefore \log(nn'n'' \ldots) = x + x' + x'' + \text{\&c.}$$

$$= \log n + \log n' + \log n'' + \text{\&c.}$$

(2) *The logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

For, if x and y be the logarithms, to the base a, of n and d,

$$\text{then,} \quad n = a^x \quad \text{and} \quad d = a^y;$$

$$\therefore \ \frac{n}{d} = \frac{a^x}{a^y} = a^{x-y},$$

$$\text{and } \therefore \ \log\frac{n}{d} = x - y = \log n - \log d.$$

(3) *The logarithm of any power or root of a quantity is found by multiplying or dividing the logarithm of that quantity by the exponent of the power or root.*

$$\text{For, if} \quad n = a^x, \quad \text{then} \quad n^p = a^{px},$$

$$\text{or} \quad \log n^p = px = p \log n.$$

$$\text{Similarly,} \quad \log n^{\frac{1}{p}} = \frac{1}{p} x = \frac{1}{p} \log n.$$

$$\text{Also} \quad \log n^{\frac{p}{q}} = \frac{p}{q} x = \frac{p}{q} \log n.$$

196. Hence, by means of the tables containing the arithmetical values of the logarithms of numbers, the operations of multiplication and division of numbers are reduced to those of addition and subtraction, and the operations of involution and evolution to those of multiplication and division.

197. In the common logarithmic tables, the decimals only of the logarithms are inserted, and the integral part, which is called the *index* or *characteristic*, is omitted, being always known from the number itself, whose logarithm is sought; for, if this number consist of two integral figures, it must be either 10 or some number between 10 and 100; and, consequently, its logarithm must be either 1 or between 1 and 2, the integral part, therefore, must be 1. In the same manner, if the number consist of three integral figures, the integral part of its logarithm must be 2, &c., so that the index, or

characteristic, is always 1 less than the number of integral figures in the number itself.

On the same principle, if the first significant figure on the left hand be in the place of tenths, the index of the logarithm will be -1, or if it be in the place of hundredths the index of the logarithm will be -2, and so on. But the negative sign is usually written over the index.

198. Hence it follows, that the logarithm of any number and the logarithm of ten times that number, differ only in the characteristic; so that the decimal parts of the logarithms of all numbers, consisting of the same figures, are the same. Thus

$$\log 185 = 2.2671717,$$

$$\log 18.5 = \log \frac{185}{10} = \log 185 - \log 10 = 1.2671717,$$

$$\log 1.85 = \log \frac{185}{100} = \log 185 - \log 100 = .2671717,$$

$$\log .185 = \log \frac{185}{1000} = \log 185 - \log 1000 = \bar{1}.2671717,$$

$$\log .0185 = \log \frac{185}{10000} = \log 185 - \log 10000 = \bar{2}.2671717,$$

and so on.

199. By means of logarithms we may easily solve the following exponential equations.

(1) $a^x = b.$

$$x \log a = \log b, \quad \text{and} \quad \therefore \quad x = \frac{\log b}{\log a}.$$

(2) $a^{b^x} = c.$

Putting y for b^x, we obtain $y = \dfrac{\log c}{\log a} = d$, suppose,

$$\text{whence } b^x = d, \quad \text{and} \quad \therefore \quad x = \frac{\log d}{\log b}.$$

INTEREST AND ANNUITIES.

200. In discussing these subjects, we shall put I for interest, P for principal, M for amount, r for the interest of £1 for one year, and n for the time for which the interest is to be calculated.

201. *To find the amount of a given sum in any time at simple interest.*

$$\text{Interest of £1 for one year} = r;$$

$$\therefore \ldots\ldots\ldots\ldots £P \ldots\ldots\ldots\ldots\ldots = Pr,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{two years} = 2Pr,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and, generally, the interest (I) of £P for n years $= Prn$;

$$\therefore M = I + P = Prn + P = P(rn + 1).$$

202. Any three of the quantities M, P, r, n being given, the fourth may be found from the equation

$$M = P(rn + 1).$$

203. Since r is the interest of £1 for one year, $100r$ is the interest of £100 for one year:

$$\therefore r = \frac{\text{rate per cent}}{100}.$$

204. *To find the amount of a given sum at compound interest.*

Let $R = 1 + r$; then PR will be the amount of P pounds at the end of one year, which then becomes the principal upon which interest is to be calculated for the ensuing year; so that, putting PR for P, we shall have, for the amount at the end of two years, PR^2: in the same manner PR^3 will be

R

the amount at the end of three years, and, generally, at the end of n years we shall have

$$M = PR^n;$$

whence, taking the logarithms, we get

$$\log M = \log P + n \log R;$$

an equation, from which, any three of the quantities involved being given, the fourth may be found.

205. *To find the amount when the principal is increased, not only by the interest, but also by some other sum at the same time.*

The amount of the original principal P in n years is PR^n, and if A be the sum which is continually added, the first A will be at interest $n-1$ years, the second $n-2$ years, &c.; whence the sum of their amounts will be

$$AR^{n-1} + AR^{n-2} + \ldots + AR^{n-n},$$

$$\text{or } A(R^{n-1} + R^{n-2} + \ldots + 1),$$

which, by taking the sum of the geometrical progression within the parenthesis, is equal to $A\dfrac{R^n-1}{R-1}$;

$$\therefore \text{ the whole amount is } PR^n + A\frac{R^n-1}{R-1}.$$

206. If $P = 0$, we have $M = A\dfrac{R^n-1}{R-1}$; which is an expression for *the amount of an annuity* (A), *at compound interest, left unpaid for* n *years.*

207. If p be *the present value* of the annuity A for n years, p must be a sum which, at compound interest for

n years, would amount to the same sum as the annuity: that is, we must have

$$pR^n = A\,\frac{R^n - 1}{R - 1}, \quad \text{and} \therefore p = A\,\frac{1 - \frac{1}{R^n}}{R - 1}.$$

DISCOUNT.

208. Let A be the sum due, P the present worth, and r, R, n as before: then

(1) *At simple interest,*

$$A = P\,(1 + rn)$$

$$\therefore\ P = \frac{A}{1 + rn},$$

$$\text{and Discount} = A - P = A - \frac{A}{1 + rn} = \frac{Arn}{1 + rn}$$

(2) *At compound interest,*

$$A = PR^n.$$

$$\therefore\ \text{Discount} = A - P = A - \frac{A}{R^n}.$$

EQUATION OF PAYMENTS.

209. *To find the equated time for the payment of two sums* P, P′, *due at the times* n, n′ *with compound interest.*

Let x be the time required; then

the interest of P for time $x - n = P\,(R^{x-n} - 1)$, Art. 204,

and the discount of P' for time $n' - x = P'\left(1 - \frac{1}{R^{n'-x}}\right)$;

$$\therefore\ P(R^{x-n}-1)=P'\left(1-\frac{1}{R^{n'-x}}\right)$$

$$\text{or}\quad P(R^{n'-n}-R^{n'-x})=P'(R^{n'-x}-1),$$

$$\text{that is,}\quad P'+PR^{n'-n}=(P+P')\,R^{n'-x},$$

an equation from which x may be obtained by the aid of logarithms.

THE END

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The following are the Subjects of Examination in Arithmetic and Algebra for the Degree of B.A. of persons not candidates for Honours.

ARITHMETIC.

Addition, Subtraction, Multiplication, Division, Reduction, Rule of Three; the same rules in Vulgar and Decimal Fractions; Practice, Simple and Compound Interest, Discount, Extraction of Square and Cube Roots*, Duodecimals.

ALGEBRA.

1. Definitions and explanations of algebraical signs and terms.

2. Addition, subtraction, multiplication, and division of simple algebraical quantities, and simple algebraical fractions.

3. Algebraical definitions of Ratio and Proportion.

4. If $a : b :: c : d$, then $ad = bc$, and the converse:

also $b : a :: d : c$,

and $a : c :: b : d$,

and $a + b : b :: c + d : d$.

* Vide *Algebra*, pp. 66 and 72.

5. If $a : b :: c : d$,
and $c : d :: e : f$,
then $a : b :: e : f$.

6. If $a : b :: c : d$,
and $b : e :: d : f$,
then $a : e :: c : f$.

7. Geometrical definition of Proportion.
(*Euc.* Book v. Def. 5.)

8. If quantities be proportional according to the algebraical definition, they are proportional according to the geometrical definition.

9. Definition of a quantity varying as another, directly, or inversely, or as two others jointly.